On Borelli's analysis concerning the deflection of falling bodies

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1 Introduction

Giovanni Alfonso Borelli can be credited to be the first person to have examined in quantitiative detail the deflection that falling bodies undergo due to the Earth's diurnal rotation.

The two primary sources are his book *De vi percussionis* (1667), and an open letter to Michelangelo Ricci (*Riposti de Gio: Alfonso Borelli Messinese* etc. 1668) in reply to criticism expressed by Stefano degli Angeli.

These sources were used by Koyré in his paper *A documentary history of the problem of fall from Kepler to Newton – De motu gravium naturaliter cadentium in hypothesi Terrae motae*, published in the *Transactions of the American Philosophical Society, 45 (4), 329–395*, in 1955.[1]

The following discussion is based on Koyré's paper. In his discussion of *De vi percussionis*, on pages 358–360, he quotes the relevant parts in the original (i.e. Latin) and also provides an English translation. Borelli's letter, however, is only given in translation, and partly paraphrased (on pages 371–376).

Another relevant publication is a paper by H.L. Burstyn, *The deflecting force of the Earth's rotation from Galileo to Newton*, published in *Annals of Science 21, 47–80*, in 1965. In his discussion of Borelli's work, he relies entirely on Koyré's paper for the source material, but he adds an interpretation of his own (more on that below). Unlike Koyré, he quotes extensively from the correspondence between Hooke and Newton; on that part of the story he has more to offer than Koyré.

In the abovementioned publications, it has passed unnoticed that *Borelli's result is inconsistent with his original assumption.* He starts with the assumption that the transverse speed be conserved. Then, by an erroneous reasoning, he arrives at a result that actually corresponds to quite a different assumption, namely that of conservation of angular momentum ! Yet, this does still not correspond

[1]This long paper was later translated into French and published as a book: *Chute des corps et mouvement de la terre – de Kepler à Newton* (Vrin, 1973).

with the modern result, because his outcome only holds if one supposes that the vertical fall is uniform (as opposed to accelerated). The purpose of this note is to dissect Borelli's reasoning and to resolve the paradox it involves.

2 De vi percussionis

A short passage in this work is devoted to the deflection of falling bodies. He first observes that an object at a certain height, turning along with the Earth, traverses a circle at a certain "transverse impetus" (*impetu transversali circulari*). Now, as the object starts falling towards the Earth's centre, it will, according to Borelli, retain its transverse impetus; it will continue to traverse equal circular distances in equal times, but since the radius of the circle becomes smaller during the fall, the traversed angles will increase [*quare si primo tempore mobile excurrit spatium DG secundo tempore ei aequali percurret spatium IH aequali ipsi DG et quia hujusmodi spatia aequalia mensurantur non in eodem, sed in divertis circulis inaequalibus fit ut angulus ACG minor sit angulo GCH, et proinde anguli praedicti successive crescunt prout distantia a centro C diminuuntur*].[2]

Borelli states very clearly here that in equal times, the object traverses equal *spatia*, i.e. spaces, distances. From an accompanying figure, it is indeed clear that these *spatia* stand for circular arcs.[3]

Now, in modern notation, we would state this as conservation of ωr (leaving the mass aside), where ω is the angular velocity, and r the radius. This azimuthal ωr being constant, the object traverses equal *spatia* in equal times, but, as the radius decreases during its fall, the object traverses ever larger angles.

The text leaves no room for an interpretation put forward by Burstyn, who claims that Borelli expresses here the principle of conservation of angular momentum (i.e. of ωr^2, again leaving the mass aside). According to Burstyn, Borelli's *impetus* is "the triple product of the mass and the tangential velocity of the body and of the radius of the curve of fall". Burstyn's interpretation would hold if Borelli had stated that the *areae* enclosed by the arcs were conserved, but Borelli speaks of *spatia*, the length of the arcs.[4]

From a historical perspective, it is also most improbable that Borelli should have come up with this principle (which we now know to be the correct one), at a stage when the development of mechanical notions was still in its infancy. It is true, of course, that Kepler had already empirically discovered the "law of areas" (his 2nd law), which, as we now know, is equivalent to conservation of angular momentum. But at the time there was no understanding of such an equivalence, and, moreover, before Newton it was not obvious that this same principle may be applicable to *terrestrial*

[2]the late Latin *diminuere* has no longer the same meaning as in classical Latin (viz. to shatter), but rather that of the classical *deminuere*, to lessen.

[3]Confusingly, however, the arcs DG and IH are not of equal length in that figure, even though he states in his text that they should. This inconstency is remedied in his later publication, discussed below.

[4]The latin *spatium*, when used in a quantitative sense, stands for 'interval', 'stretch', or 'distance'. Only in the vaguer, broader sense, may it mean 'room' or 'space'.

bodies; there was still a sense that the 'sublunar' obeys different laws than the celestial spheres. Indeed, Borelli nowhere mentions such an analogy, nor does he refer to Kepler's laws.

3 Borelli's open letter

We follow here the translation provided by Koyré (Transactions, p. 371–374).

Regarding the falling motion of a stone dropped from a tower, Borelli first reiterates his guiding principle:

"[I] believe [that] in whatever place of the descent it finds itself, it must conserve the same degree of transverse velocity, and consequently traverse equal spaces in equal times on all the circles that it traverses" (p. 371).

(Notice, again, that Borelli assumes the transverse speed, not angular momentum, to be conserved – contradicting Burstyn's claim.) The situation he now considers is the following:

"Coming now to our case, let C [Fig. 1] be the center of the Earth , the circumference of the Equator be EH, and let AH be the height of the tower [. . .] and let it be admitted that AE, together with the semidiameter EC, perform a uniform circular motion along the circumference EH; let us now drop a stone from the summit A." (p. 373)

He then goes on to estimate the magnitude of the deflection:

"The tower AE is 240 feet high, and it is supposed to circumgyrate with the terrestrial semidiameter CE on the arc EH of one minute of the Equator in such a manner that the tower moves to the site HM and that in this time a ball of chalk falling from the top A arrives at the Earth with two motions, i.e. with the transverse motion, of which the uniform *impetus* is measured by the arc AM and with the descending *impetus* on the perpendicular AE. I say that in this case the ball will not fall precisely upon the lowest place H of the perpendicular to the horizon HM traced on the face of the tower, but that falling along the line MO, it will outrun it somewhat, [and that] the arc EO will have to be equal to AM and thus greater than EH. I must show now that the deviation HO, on account of its smallness, cannot be observed, because as the terrestrial semidiameter CA is supposed to be of 23,367,468 Roman feet (antique) and the tower EA of having a height of 240 of the same feet, the proportion of the terrestrial arc EH of a minute of the Equator to the arc AM traversed by the top of the tower, will be the same as that of CE to AC; now this transit is made in 4″ of the hour; therefore, admitting the arc HE to be of 6,797 1/2 Roman feet, the arc AM, or thus EO [will have] 6,797 48/120 feet and consequently the excess HO will be 8/120 foot, which is 8/12 inch, and thus less than an inch."

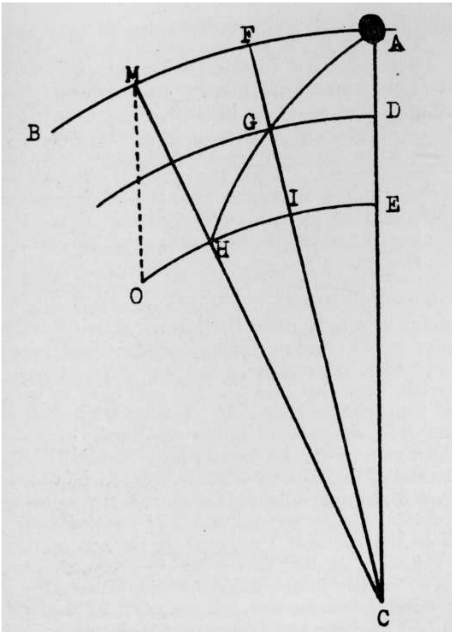

Figure 1: From Borelli's letter.

His result amounts to 2.0 cm (one Roman foot being 29.6 cm). His reasoning can be rendered in a simpler way as follows. Starting from the presumed equality of AM and EO,

$$AM = EO\,,$$

and assuming the fall to take an amount of time T (which Borelli takes to be 4 seconds), we have, with the Earth's angular velocity Ω,

$$AM = \Omega(AE + EC)T\,; \qquad EH = \Omega\, EC\, T\,.$$

Hence

$$HO = EO - EH = AM - EH = \Omega\, AE\, T\,. \tag{1}$$

This amounts to 0.07 foot, the value Borelli arrived at.

As Burstyn notes correctly, this is larger than the modern value, which would be $\frac{2}{3}\Omega\, AE\, T$ (see below). Whence comes the difference? According to Burstyn, Borelli "has implicitly assumed that the fall of the body is uniform", and argues that an accelerated motion would have led to a smaller value. The problem is, however, that Borelli's reasoning does not seem to depend on the nature of the fall, but only on its *duration*.

The conundrum becomes still larger by the fact that Borelli started out with the incorrect assumption of a "uniform transverse impetus"; he should instead have applied conservation of angular momentum. The latter principle is what gives the factor "2" in the Coriolis force in the term governing the deflection at hand. So, this correction would make Borelli's value (1), which already *is* too large, even larger !

3.1 The Coriolis force

The eastward deflection is governed by the Coriolis force term

$$\frac{\partial u}{\partial t} = -2\Omega\cos(\phi)w\,.$$

At the equator (Borelli's case), we have $\cos\phi = 1$. Double integration in time then gives,

$$\Delta = -2\Omega \int_0^T dt \int_0^t dt'\, w\,. \tag{2}$$

where Δ is the eastward deflection (i.e. HO in Borelli's notation).

It is now an interesting exercise to calculate the deflection assuming a *uniform* vertical motion, i.e. to take w to be constant: $w = -h/T$, h the height of the tower. This gives

$$\Delta_{unif} = \Omega h T \qquad \text{(for a \textit{uniform} fall)}\,. \tag{3}$$

This is the *same* as Borelli's (1)! This is the all the more remarkable, as the Coriolis term is based on conservation of angular momentum (ωr^2), not of transverse speed (ωr). And the *latter* is what Borelli assumed !

To resolve the contradiction, let us reconsider Borelli's argument. He begins with the assumption that the stone's transverse speed, $v = \Omega(CE + EA)$, is conserved. From this he infers that the stone will traverse an arc of length AM, irrespective of whether it stays at the top or falls downward. Now, had the object begun an eastward trajectory at the *foot* of the tower (E), and moved from there at speed v, then it would indeed have ended up at O after time T, since EO is equal to AM. On the other hand, had the object stayed at the top, it would have ended up in M. Now, the stone does in fact neither of these two things, since it starts at the top A, but ends up at the Earth's surface. Assuming a uniform vertical fall, we should therefore take the *mean* height, i.e. D, as a starting point for drawing the arc. But this means that the stone reaches the eastward position given by the *middle* of the dashed line OM, which amounts to an eastward deflection of precisely *half* the distance HO ! In other words, had Borelli followed through his argument in a consistent way, he would have found the deflection

$$\frac{1}{2}\Omega\, AE\, T\,.$$

instead of (1).

Replacing now conservation of transverse speed by conservation of angular momentum, we find twice the previous result, i.e. $\Omega\, AE\, T$, which is consistent with (3).

Turning, finally, to the correct solution, as found by Laplace, we use again (2), but now take the correct expression for w, the one that corresponds to a uniformly accelerated fall: $w = -gt'$. Hence

$$\Delta_{acc} = \frac{1}{3} g \Omega T^3 \, .$$

Or, since $h = \frac{1}{2} g T^2$,

$$\Delta_{acc} = \frac{2}{3} h \Omega T \, . \tag{4}$$

By comparing (3) and (4), we see that the difference between a uniform and a uniformly accelerated vertical motion amounts to a factor of 2/3.

In summary, we may distinguish four cases:

	conservation of transverse velocity	conservation of angular momentum
uniform fall	$\frac{1}{2}\Omega h T$	$\Omega h T$
uniformly accelerated fall	$\frac{1}{3}\Omega h T$	$\frac{2}{3}\Omega h T$

The correct answer, of course, is that of the lower right corner. Based on his assumptions, Borelli *should* have found the result stated in the upper left corner, but, by an erroneous reasoning, he ended up with the expression of the upper right corner.

3.2 Kepler's problem

The most rigorous way to treat the problem is to conceive it as a "Kepler problem", which in this case amounts to determining the trajectory of a satellite in a given gravity field, for a given initial position and velocity of the satellite.

This means we have to solve the following equations

$$\frac{dr}{dt} = w \tag{5}$$

$$\frac{dw}{dt} = -\frac{GM}{r^2} + \frac{L^2}{r^3} \tag{6}$$

$$\frac{d\Theta}{dt} = L/r^2 \, . \tag{7}$$

(The equations have here been written in a form that is convenient for numerically solving them.) The third equation expresses conservation of angular momentum, L being constant.

There are two slight differences with respect to Laplace's solution, (4). First, gravity g is not taken constant, but is now given by $-GM/r^2$. This turns out to be a very minute effect. More important is the second difference: the vertical movement is now being calculated, instead of being imposed as if it were identical to a fall on a non-rotating Earth. The latter procedure was followed above (and by Laplace, too), when we prescribed w. This is not completely exact. After all, the kinetic energy the stone has gained when it reaches the Earth's surface should be the same on a rotating and a non-rotating Earth (since the same amount of potential energy is being converted in both cases). Now, in the former case, part of this gain goes into a (small) eastward velocity, associated with the eastward deflection, which implies that the terminal vertical velocity must be slightly smaller than on a non-rotating Earth.

These two aspects are calculated exactly by solving the "Kepler problem". Specifically, the outcome is as follows. The eastward deflection is found to be 1.3140 cm, which is slightly larger than what (4) yields (i.e. 1.3117 cm). The terminal vertical velocity is 37.3017 m/s, which is slightly smaller than for the non-rotating case, 37.3661 m/s (or the virtually identical 37.3663 m/s in Laplace's case, where gravity is taken constant). In other words, the vertical velocity is decreased by 0.17% with respect to Laplace's expression, which gives slightly more time for the eastward deflection to build up, and which is therefore 0.17% larger than in Laplace's expression.