Destabilizing a seiche with an movable dam.

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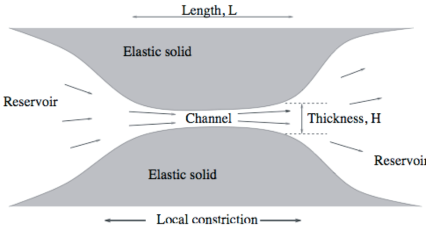
GFD Program

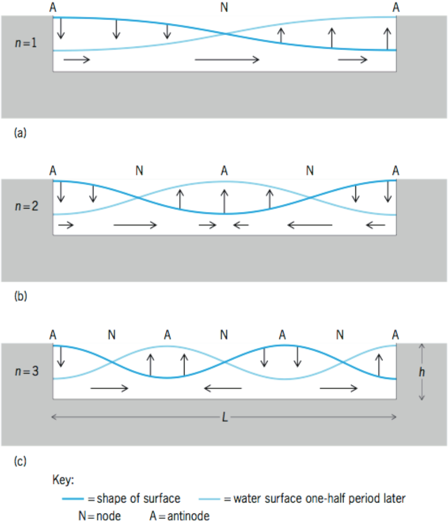
Volcanic tremors and clarinet



[Backus 1963]

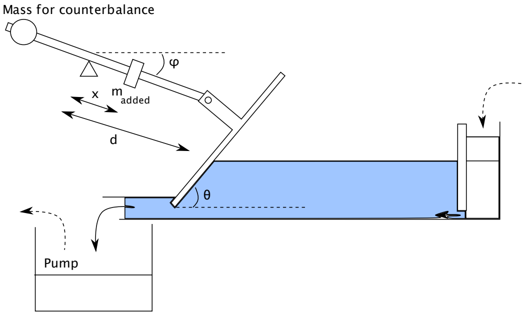
Oscillating reed or rock excitating and interacting with standing waves in the adjacent reservoir.

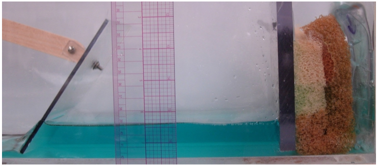
Seiches in a box : standing waves.



$$T = \frac{2L}{n\sqrt{gh}}$$

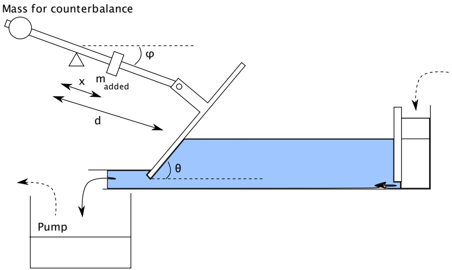
'Water' Clarinet : movable dam





A shallow water model

Governing equations



Conservation of angular momentum :

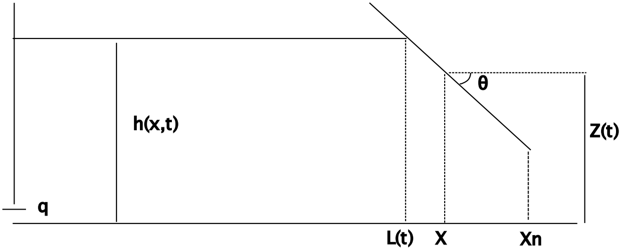
$$J\ddot{\phi} = -mgd\cos(\phi) + Wd\cos(\phi)\cos\theta \int_L^{X_N} p dx \qquad (1)$$

$$J = (M+m)d^2$$

M : effective mass and mass m excess equivalent mass placed on the paddle : $m_{\text{added}} x^2 = md^2$

A shallow water model

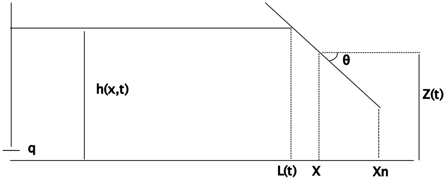
Governing equations



ϕ small angle $\cos(\phi) \approx 1$ and $\dot{Z} \approx d\dot{\phi}$

$$(M + m)\ddot{Z} = -mg + W \cos\theta \int_L^{X_N} p dx$$

Governing equations



Equations for $0 < x < L$:

$$h_t + (hu)_x = 0$$

$$u_t + uu_x = -gh_x$$

Boundary conditions :

$[hu]_{x=0} = q$

$h(x = L) = h_L = Z + (X - L)\tan\theta$

Equations for $x > L$:

$$h = Z(t) + (X - x)\tan\theta$$

$$h_t = \dot{Z} = -(hu)_x$$

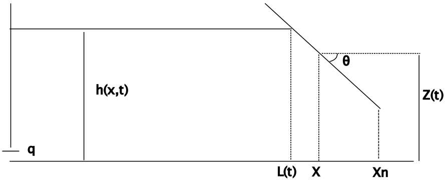
$$u_t + uu_x = -\frac{p_x}{\rho}$$

Boundary conditions :

$u(x \to L^-) = u_L$

$h(x \to L^-) = h_L = Z + (X - L)\tan\theta$

Scaling and non dimensional parameters



Scaling of distances

- Reservoir : $\hat{x} = \frac{x}{X}$
- Paddle : $x = (X_N - X)\xi + X$ with $l < \xi < 1$ ie $\hat{x} = R\xi + 1$ where $R = \frac{X_N - X}{X}$

Scaling and non dimensional parameters

$$\hat{h} = \frac{h}{(X_N - X)\tan\theta} \qquad \hat{u} = \frac{u}{\sqrt{g(X_N - X)\tan\theta}}$$

$$\hat{t} = \frac{t}{\frac{X}{\sqrt{g(X_N - X)\tan\theta}}} \qquad \hat{p} = \frac{p}{\rho g(X_N - X)\tan\theta}$$

Equations for $0 < x < L$:

q=hu
$\hat{h}_{\hat{t}} + \hat{q}_{\hat{x}} = 0$
$\hat{q}_{\hat{t}} + (\frac{\hat{q}^2}{\hat{h}})_{\hat{x}} = -\hat{h}\hat{h}_{\hat{x}}$

Boundary conditions :
$[\hat{h}\hat{u}]_{\hat{x}=0} = \hat{q}$
$\hat{h}_{\hat{L}} = h(\hat{x} = 1 + Rl)$

Equations for $x > L$:

$\hat{h} = \hat{Z} - \xi$

$\hat{h}_t = \dot{\hat{Z}} = -(\hat{h}\hat{u})_{\hat{x}}$
$\hat{p}_\xi = -\hat{u}\hat{u}_\xi - R\hat{u}_{\hat{t}}$

Boundary conditions :
$\hat{u}_{\hat{L}} = u(\hat{x} = 1 + Rl)$
$\hat{h}_{\hat{L}} = h(\hat{x} = 1 + Rl)$
$\hat{u} = \frac{\hat{q}_L}{\hat{Z}-\xi} - R\frac{\xi - l}{\hat{Z}-\xi}$

The equation for the paddle becomes :

$$I\ddot{\hat{Z}} = -1 + \mu \int_l^1 p d\xi$$

The system has 5 parameters :

- $I = \dfrac{m+M}{m} R^2 \tan^2\theta$ the inertia term,
- $\mu = \dfrac{\rho W (X_N - X)^2 \sin\theta}{m}$ the ratio mass water/mass on paddle
- $R = \dfrac{X_N - X}{X}$
- $\hat{Q} = \dfrac{q}{\sqrt{g}((X_N - X)\tan\theta)^{3/2}}$ the flow rate

Approximation R small

Equations for $x > L$:

$$\hat{u} = \frac{\hat{q}_L}{\hat{Z} - \xi} - R\frac{\xi - l}{\hat{Z} - \xi}$$

$$\hat{p}_\xi = -\hat{u}\hat{u}_\xi - R\hat{u}_{\hat{t}}$$

$$I\ddot{\hat{Z}} = -1 + \mu \int_l^1 p d\xi$$

Approximation R small : $u \approx \frac{q_L}{Z-\xi} = \frac{q_L}{h}$ and $p_\xi \approx -uu_\xi$

ie Bernoulli is verified under the paddle.

$$p + 1/2u^2 = B = \text{constant} = h_N + 1/2u_N^2 = h_L + 1/2u_L^2$$

Compact form using the ratio of heights $\alpha = \frac{h_L}{h_N}$:

$$\boxed{\frac{q_L^2}{2h_N^3} = \frac{Fr_N^2}{2} = \frac{\alpha^2}{\alpha + 1}} > 1$$

By integrating to get the pressure force, the equation of motion for the paddle becomes :

$$I\ddot{\hat{Z}} = -1 + \frac{q_L^{4/3}\mu}{2^{2/3}} F(\alpha)$$

where

$$F(\alpha) = \frac{(\alpha - 1)(\alpha^2 + 1)}{\alpha^{4/3}(\alpha + 1)^{1/3}} \quad (2)$$

Then the steady state (Q_L, H_L, H_N) is :

$$F(\alpha_0) = \frac{2^{2/3}}{Q_L^{4/3}\mu} \quad (3)$$

Linear stability analysis

$$h_t + q_x = 0 \tag{4}$$

$$q_t + (\frac{q^2}{h})_x = -hh_x \tag{5}$$

Linearization :

$$h = H + h' \quad h_L = H_L + h'_L \text{ and } h_N = H_N + Z'$$
$$q = Q + q'$$
$$Z = Z_0 + Z'$$

where $H(= H_L)$ and $Q(= Q_L)$ are the stationary state.

$$h'_t + q'_x = 0 \tag{6}$$

$$q'_t + \frac{2Q}{H} q'_x - \frac{Q^2}{H} h'_x = -Hh'_x \tag{7}$$

$$\tag{8}$$

$$h'_t + q'_x = 0 \tag{9}$$

$$q'_t + \frac{2Q}{H} q'_x - \frac{Q^2}{H} h'_x = -H h'_x \tag{10}$$

which can be combined to give :

$$\left(\partial_t + \frac{Q}{H}\partial_x\right)^2 h' = H h'_{xx}$$

We seek for $q' = \tilde{q} e^{-i\omega t}$ and $h' = \tilde{h} e^{-i\omega t}$ which gives :

$$\left(-i\omega + \frac{Q}{H}\partial_x\right)^2 \tilde{h} = H \tilde{h}_{xx}$$

We seek solution of the from $e^{\lambda x}$.

$$(-i\omega + \frac{Q}{H}\lambda)^2 - H\lambda^2 = 0$$

$$\lambda_1 = \frac{i\omega}{\frac{Q}{H} - \sqrt{H}} \quad \text{and} \quad \lambda_2 = \frac{i\omega}{\frac{Q}{H} + \sqrt{H}}$$

Thus $\tilde{q} = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}$ and $\tilde{h} = A_3 e^{\lambda_1 x} + A_4 e^{\lambda_2 x}$

A_1 can be chosen arbitrary to 1.

Boundary conditions q'=0 at x=0 so $A_2 = -1$.

Then $i\omega \tilde{h} = \tilde{q}_x$ gives $A_3 = \frac{1}{\frac{Q}{H}+\sqrt{H}}$ and $A_4 = -\frac{1}{\frac{Q}{H}-\sqrt{H}}$
Variables $\beta = \frac{Q}{H^{3/2}}$ and $\tilde{\Omega} = \frac{\Omega}{(1-\beta^2)}$where $\Omega = \frac{\omega}{\sqrt{H}}$, q'and h' are written :

$$q' = e^{-i\tilde{\Omega}\beta x} 2i \sin(\tilde{\Omega}x) e^{-i\omega t}$$

$$h' = e^{-i\tilde{\Omega}\beta x} \frac{2\tilde{\Omega}}{\omega} \left[\cos(\tilde{\Omega}x) - \beta i \sin(\tilde{\Omega}x)\right] e^{-i\omega \hat{t}}$$

Paddle and Bernoulli linearized equations :

$$-\omega^2 I Z' = \frac{4}{3}\frac{\mu}{2^{2/3}} Q^{1/3} q'_L F(\alpha_0) + \frac{\mu}{H_N}\frac{F'(\alpha_0)}{F(\alpha_0)}(h'_L - \alpha_0 Z') \qquad (11)$$

$$Q q'_L - h'_L H_N^2 \frac{\alpha_0(\alpha_0+2)}{(\alpha_0+1)^2} = \frac{H_N^2 \alpha_0^2 Z'(2\alpha_0+1)}{(\alpha_0+1)^2} \qquad (12)$$

where $F'(\alpha_0) = \frac{dF}{d\alpha}(\alpha_0)$

Matching conditions :

$$q'_L = q'(x = 1 + R\ l(t), t) \approx q'(x = 1, t)$$

and also $h'_L \approx h'(x = 1)$

The Bernoulli equation combined with the equation of the paddle and using the matching conditions gives an equation allowing to determine ω with :

$$D(\omega; \alpha_0, I, \mu, Q, H) = 0$$

Solutions $\omega = \omega_r + i\ \omega_i$ found numerically.

Physically : destabilization of a seiche mode maintained by the matching of time scales.

Approximation small flow rate : $Q \ll 1$ or $\alpha_0 \gg 1$

In the hypothesis where the flow rate small ie $\alpha_0 \gg 1$, $F(\alpha_0) \sim \alpha_0^{4/3}$ and by keeping the leading order terms in α_0 in $D(\omega) = 0$, it gives

$$\Omega \approx n\pi$$

ie in dimensional variables :

$$\omega = n\pi \frac{\sqrt{gH_L}}{X}$$

By looking at $\Omega = n\pi + \gamma$, we have the growth rate

$$\gamma = \frac{i(1 - \frac{n^2\pi^2 I}{4\alpha_0\mu^2})}{\frac{1}{3}\alpha_0\sqrt{2}(-1 + \frac{1}{\mu} + \frac{3n^2\pi^2 I}{4\mu^2\alpha_0})} \qquad (13)$$

Numerical application :

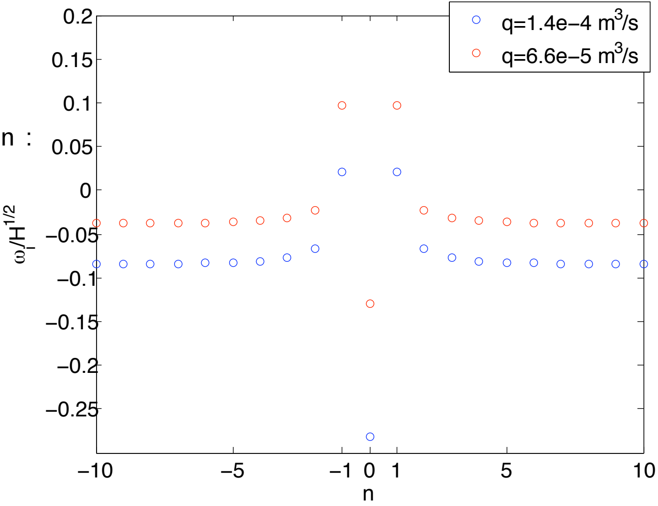
m_{added}=3.1g

x=5cm

X=18cm

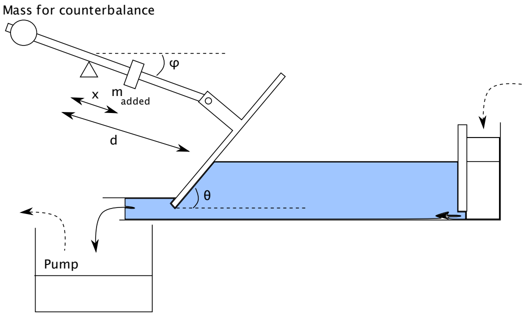
$\theta = 60^{\circ}$

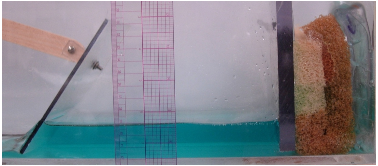
$H_L \approx 2cm$



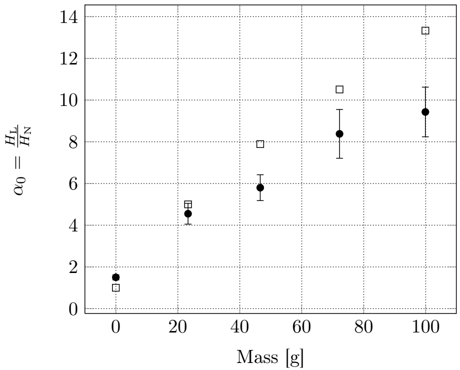
Experiments

Experimental setup



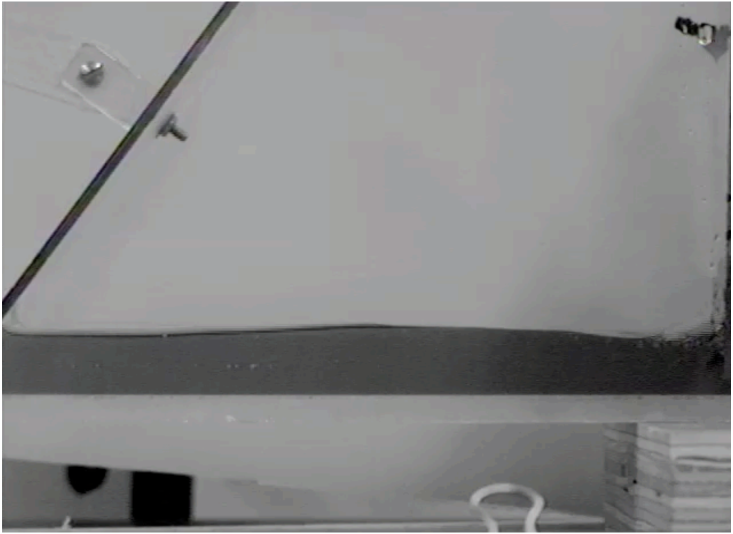


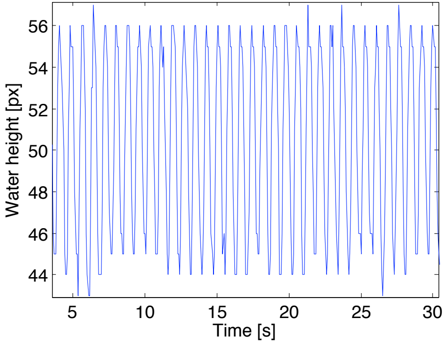
Steady state

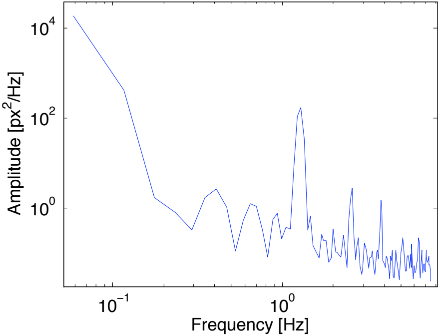


Comparison between experimental points (dots) and numerical calculation which inverses $F(\alpha_0) = \frac{2^{2/3}}{Q_L^{4/3}\mu}$ (squares). $\theta = 30^\circ$ q=$1.6e-4\ m^3/s$

Instability of the seiche



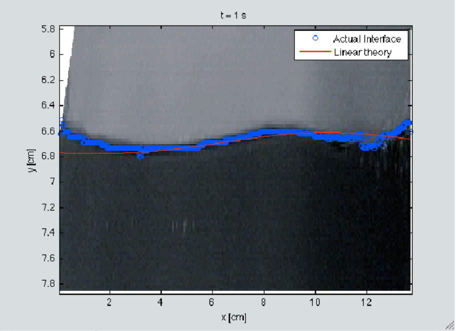




Time series for m_{added}=3.1g at 5cm of the pivot

Water height : comparison between linear theory and the experiment :

$$h' = e^{-i(\omega t - \tilde{\Omega}\beta x)} \frac{2\tilde{\Omega}}{\omega} cos(\tilde{\Omega}x) \quad \text{for } n = 1$$

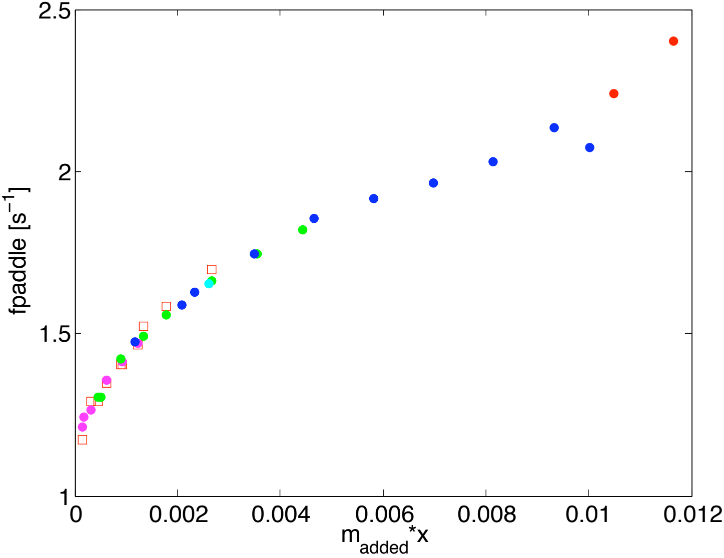


Experiment with m_{added}=3.1g at 5 cm of the pivot.

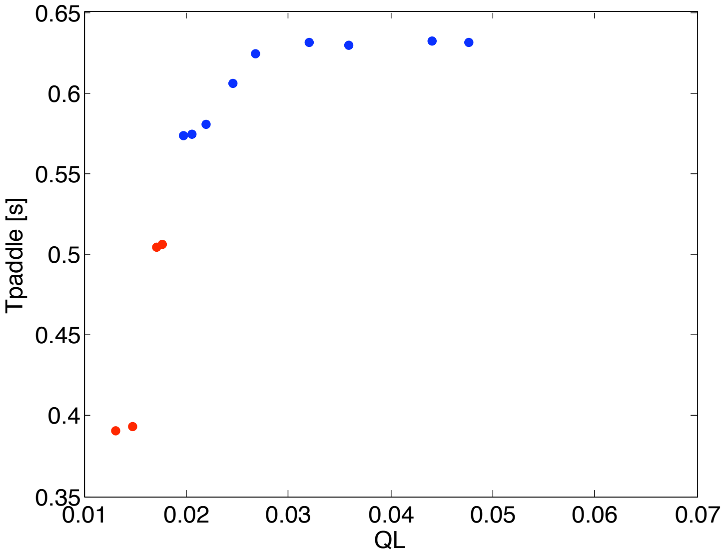
Mode 2 dominant ?

Variation of parameters

Variation of m_{added} at q fixed. ($q = 1.42 \times 10^{-4} m^3/s$)

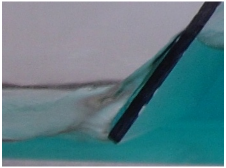


Variation of q at m_{added} fixed. (m_{added}=8.9 g and x=25 cm)



Limits of the model

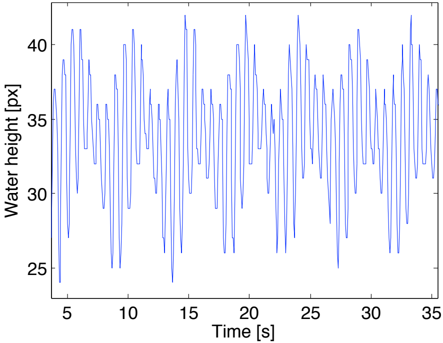
- Collision with the bottom.

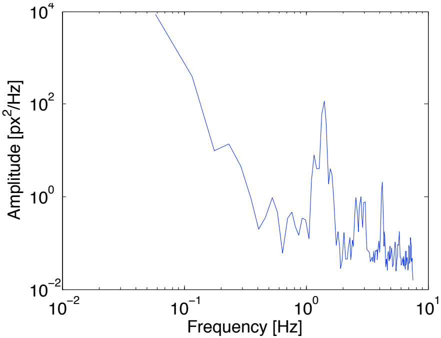


- Side effects : flow around the paddle
- Friction in the hinge.
- Viscosity or surface tension not included.
- Validity of the shallow water hypothesis.

Evidence of non-linearities

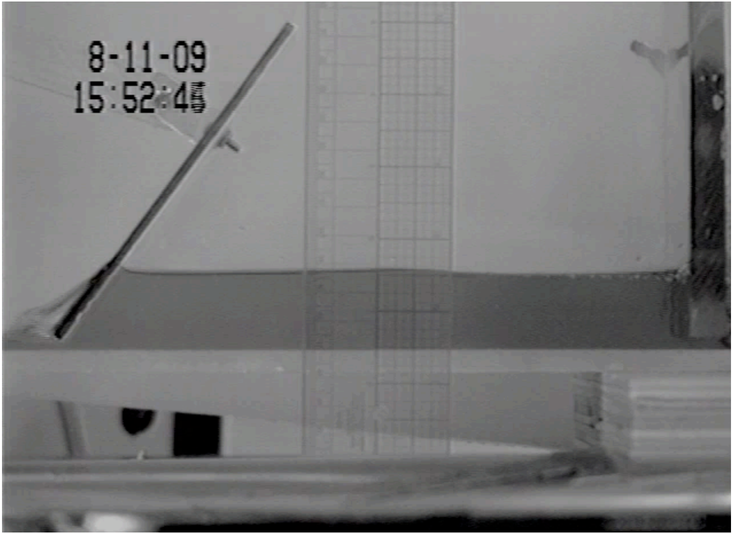
Non-linear interaction of modes



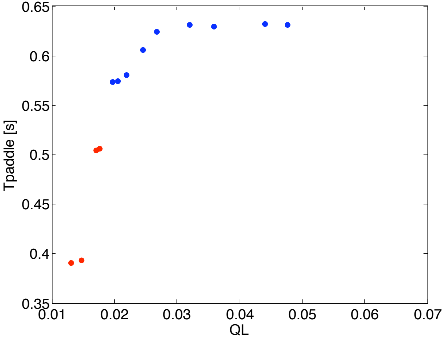


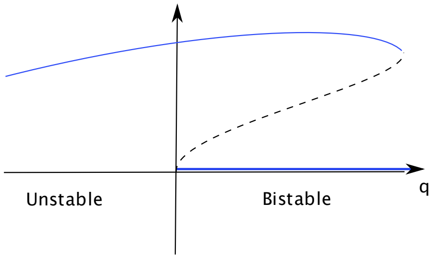
Time series for m_{added}=3.1g at 30cm of the pivot

Evidence of non-linearities
Bistabilité



Subcritical bifurcation ?





red : unstable

blue :bistable

Conclusion

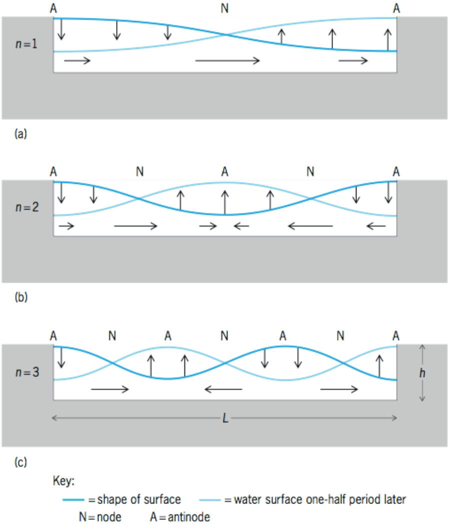
To remember

- ✓ Highlight of the instability.
- ✓ Agreement between theory and observation of seiche modes

To be done

- ✓ Non-linear theory
- ✓ Role of side effects

Seiches in a box : standing waves.



$$T = \frac{2L}{n\sqrt{gh}}$$