Open Channel Hydraulic

Julien Chauchat

Associate Professor - Grenoble INP / ENSE3 - LEGI UMR 5519

julien.chauchat@grenoble-inp.fr

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Natural / artificial channels

River

Irrigation channel

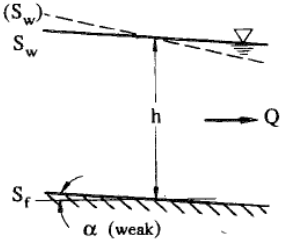
Wastewater treatment plant

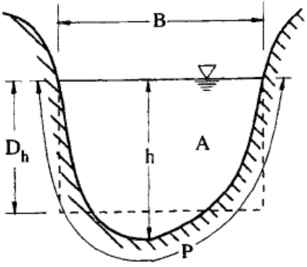
Storm water overflow sewer

Channels geometries

Water surface slope
piezometric slope

Bed slope

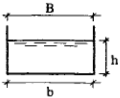
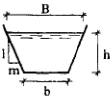
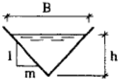
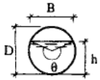
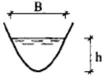




Definitions:

- Cross-section (CS): plane normal to the flow direction
- A: *Wetted surface* = portion of the cross section occupied by the liquid
- P: *Wetted perimeter* = length of contact line between liquid and bed and banks
- $R_h = \dfrac{A}{P}$: *Hydraulic radius* = reference length of the CS
- B: *Width or top width* = width of the channel at the free surface
- $D_h = \dfrac{A}{B}$: *Hydraulic depth* = water depth of an equivalent rectangular CS
- h: *Water depth* = maximum depth in the CS

Channels geometries

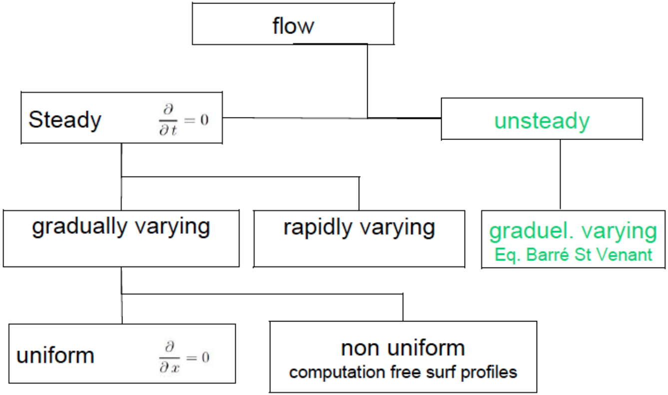
	 Rectangle	 Trapezoid	 Triangle	 Circle	 Parabola
Section A	$b\,h$	$(b+mh)h$	mh^2	$\frac{1}{8}(\theta-\sin\theta)\,D^2$	$\frac{2}{3}B\,h$
Wetted perimeter P	$b+2h$	$b+2h\sqrt{1+m^2}$	$2h\sqrt{1+m^2}$	$\frac{1}{2}\,\theta\,D$	$B+\frac{8}{3}\,\frac{h^2}{B}$ *
Hydraulic radius R_h	$\frac{b\,h}{b+2h}$	$\frac{(b+mh)\,h}{b+2h\sqrt{1+m^2}}$	$\frac{mh}{2\sqrt{1+m^2}}$	$\frac{1}{4}\left[1-\frac{\sin\theta}{\theta}\right]D$	$\frac{2B^2h}{3B^2+8h^2}$ *
Width B	b	$b+2mh$	$2mh$	$(\sin\theta/2)\,D$ or $2\sqrt{h\,(D-h)}$	$\frac{3}{2}\,\frac{A}{h}$
Hydraulic depth D_h	h	$\frac{(b+mh)\,h}{b+2mh}$	$\frac{1}{2}h$	$\left[\frac{\theta\angle\sin\theta}{\sin\theta/2}\right]\frac{D}{8}$	$\frac{2}{3}h$

From Graf and Altinakar (1998)

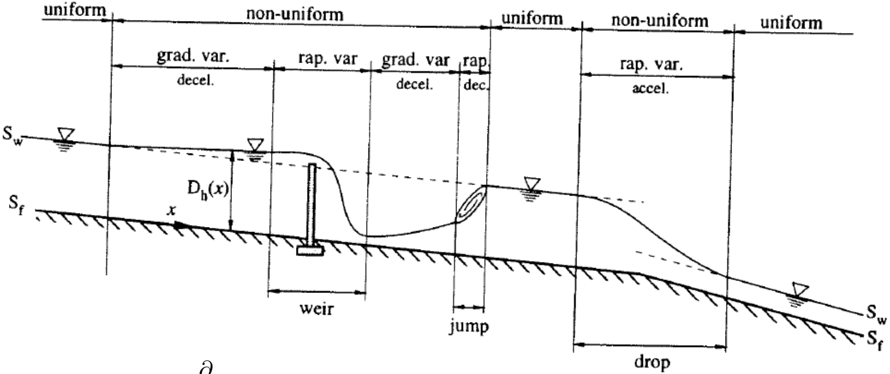
Type of flows

- Flow with a free surface (air/water interface) on which the pressure is at the atmospheric value
- The flow is mostly driven by gravity due to the inclination of the bed (and not due to a pressure drop like in closed conduit flows)

Classification:



Type of steady flows



- Uniform flows: $\dfrac{\partial}{\partial x} = 0$ and $S_f = S_w$
- Non-uniform flows: $D_h(x)$ varies and $S_f \neq S_w$
  - Gradually varied flows: "slow variation" / quasi-uniform flow
  - Rapidly varied flows: abrupt change of $D_h(x)$ like at a singularity (weir, gate,...) or a Hydraulic jump

Navier-Stokes equations

Navier-Stokes equation (incompressible)

Mass conservation

$$\overrightarrow{\nabla}.(\overrightarrow{u}) = 0$$

Momentum balance equation

$$\rho \frac{\partial \overrightarrow{u}}{\partial t} + \underbrace{\rho\ \overrightarrow{\nabla}.(\overrightarrow{u} \otimes \overrightarrow{u})}_{Inertia} = \underbrace{-\ \overrightarrow{\nabla} p}_{Pressure} + \underbrace{\eta\ \overrightarrow{\nabla}^2 \overrightarrow{u}}_{Viscous\ stress} + \underbrace{\rho\ \overrightarrow{g}}_{Gravity}$$

Dimensional analysis (1/5)

Let's consider

- a length scale: L_0
- a velocity scale: U_0

based on which we can deduce

- a time scale: $T_0 = \dfrac{L_0}{U_0}$
- a pressure scale: $P_0 = \rho \ (U_0)^2$

With these scales we can make the variables dimensionless:

- position : $\overrightarrow{\bar{x}} = \dfrac{\overrightarrow{x}}{L_0}$
- temps : $\bar{t} = \dfrac{t \ U_0}{L_0}$
- vitesse : $\overrightarrow{\bar{u}} = \dfrac{\overrightarrow{u}}{U_0}$
- pression : $\bar{p} = \dfrac{p}{\rho U_0^2}$

Dimensional analysis (2/5)

We now substitute these dimensionless variables into the Navier-Stokes equations:

Mass conservation

$$\frac{U_0}{L_0}\overrightarrow{\overline{\nabla}}.\left(\overrightarrow{\overline{u}}\right)=0 \quad \longrightarrow \quad \overrightarrow{\overline{\nabla}}.\left(\overrightarrow{\overline{u}}\right)=0$$

Momentum balance

$$\rho\frac{U_0^2}{L_0}\left[\frac{\partial\overrightarrow{\overline{u}}}{\partial\bar{t}}+\overrightarrow{\overline{\nabla}}.\left(\overrightarrow{\overline{u}}\otimes\overrightarrow{\overline{u}}\right)\right]=-\frac{\rho\, U_0^2}{L_0}\;\overrightarrow{\overline{\nabla}}\bar{p}+\frac{U_0}{L_0^2}\eta\;\overrightarrow{\overline{\nabla}}^2\overrightarrow{\overline{u}}+\rho\;\parallel\overrightarrow{g}\parallel\;\frac{\overrightarrow{g}}{\parallel\overrightarrow{g}\parallel}$$

dividing by $\rho\dfrac{U_0^2}{L_0}$:

$$\left[\frac{\partial\overrightarrow{\overline{u}}}{\partial\bar{t}}+\overrightarrow{\overline{\nabla}}.\left(\overrightarrow{\overline{u}}\otimes\overrightarrow{\overline{u}}\right)\right]=-\overrightarrow{\overline{\nabla}}\bar{p}+\underbrace{\frac{\eta\; U_0\; L_0}{\rho\; L_0^2\; U_0^2}}_{\dfrac{\eta}{\rho\; U_0\; L_0}}\;\overrightarrow{\overline{\nabla}}^2\overrightarrow{\overline{u}}+\underbrace{\frac{L_0\;\parallel\overrightarrow{g}\parallel}{U_0^2}}_{\dfrac{L_0\;\parallel\overrightarrow{g}\parallel}{U_0^2}}\;\frac{\overrightarrow{g}}{\parallel\overrightarrow{g}\parallel}$$

Dimensional analysis (3/5)

These two numbers are nothing else but:

- the Reynolds number: $\boxed{Re = \frac{\rho\ U_0\ L_0}{\eta}}$
- the Froude number: $\boxed{Fr^2 = \frac{U_0^2}{L_0\ \parallel \overrightarrow{g} \parallel}}$

The dimensionless momentum balance equation can be rewritten as:

$$\left[\frac{\partial \overrightarrow{\bar{u}}}{\partial \bar{t}} + \overrightarrow{\bar{\nabla}}.\left(\overrightarrow{\bar{u}} \otimes \overrightarrow{\bar{u}}\right)\right] = -\overrightarrow{\bar{\nabla}}\bar{p} + \frac{1}{Re}\ \bar{\Delta}\overrightarrow{\bar{u}} + \frac{1}{Fr^2}\ \overrightarrow{\bar{g}}$$

with $\overrightarrow{\bar{g}} = \frac{\overrightarrow{g}}{\parallel \overrightarrow{g} \parallel}$

Dimensional analysis (4/5)

Now let's take some characteristic scales of the problem:

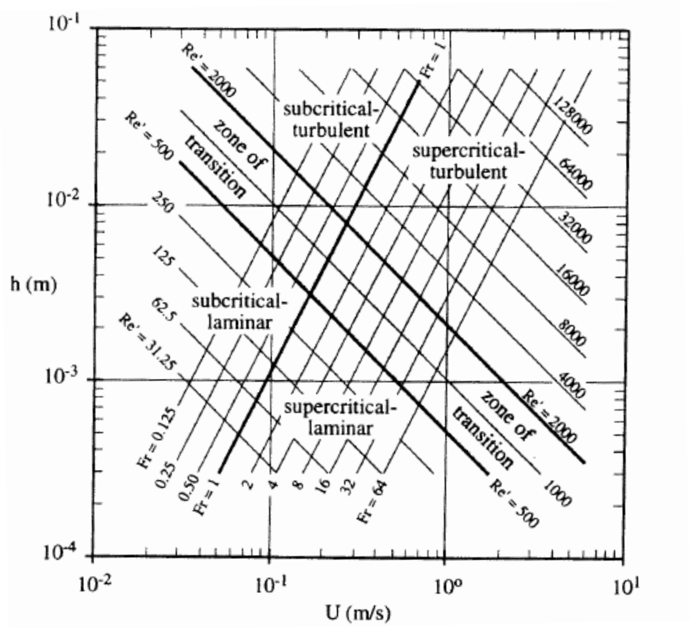
- Length scale = water depth : $L_0 = h \approx 0.1 - 10 \ m$
- Velocity scale = mean velocity : $U_0 = \dfrac{Q}{A} \approx 0.1 - 10 \ m/s$

and compute the order of magnitude of these two dimensionless numbers:

- Reynolds: $Re = \dfrac{\rho \ U_0 \ L_0}{\eta} = \dfrac{10^3(10^{-1} - 10^1) \ (10^{-1} - 10^1)}{10^{-3}} = 10^4 - 10^8$
- Froude: $Fr = \dfrac{U_0}{\sqrt{L_0 \ \| \overrightarrow{g} \|}} = \dfrac{(10^{-1} - 10^1)}{\sqrt{(10^{-1} - 10^1) \ 10}} = 10^{-2} - 10$

Conclusion: In natural systems, the flow is always fully turbulent ($Re > 2 \ 10^3$) and can be *sub- or supercritical*.

Flow regimes



From Graf and Altinakar (1998)

Dimensional analysis (5/5)

The order of magnitude of the different terms in the momentum equation can be estimated as follows:

$$\left[\frac{\partial \vec{\bar{u}}}{\partial \bar{t}} + \vec{\bar{\nabla}}.\left(\vec{\bar{u}} \otimes \vec{\bar{u}}\right)\right] = -\vec{\bar{\nabla}}\bar{p} + \underbrace{\frac{1}{Re}\,\bar{\Delta}\vec{\bar{u}}}_{O(10^{-8}-10^{-4})} + \underbrace{\frac{1}{Fr^2}\,\vec{\bar{g}}}_{O(10^{-2}-10^{4})}$$

The viscous stress tensor is negligible but due to turbulence the inertial term give rise to a turbulent stress, the so-called Reynolds stress. Very quickly (you will see that in more details in the mixing part of this lecture) the Reynolds stress is obtained by introducing the Reynolds decomposition

$$<\vec{u}> = \frac{1}{\tau}\int_t^{t+\tau} \vec{u}\,dt$$

$$\vec{u} = <\vec{u}> + \vec{u'}$$

$$<\vec{u}> = << \vec{u} >>$$

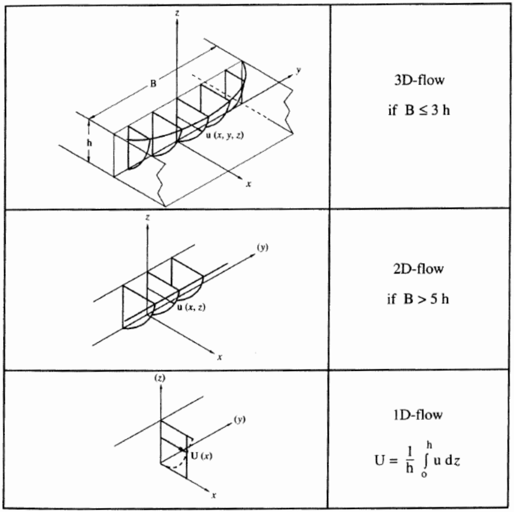
$$<\vec{u'}> = 0$$

and applying it to the inertia term:

$$\left\langle \vec{u} \otimes \vec{u} \right\rangle = \left\langle \left(<\vec{u}> + \vec{u'}\right) \otimes \left(<\vec{u}> + \vec{u'}\right) \right\rangle$$

$$\left\langle \vec{u} \otimes \vec{u} \right\rangle = <\vec{u}> \otimes <\vec{u}> + \underbrace{\left\langle \vec{u'} \otimes \vec{u'} \right\rangle}_{-Reynolds\ stress\ tensor}$$

Transverse velocity profile



In Hydraulic engineering we assume that the flow one-dimensional:

$$U = \frac{1}{A}\int\int_A \vec{u}(x,y,z).\vec{x}\,dA$$

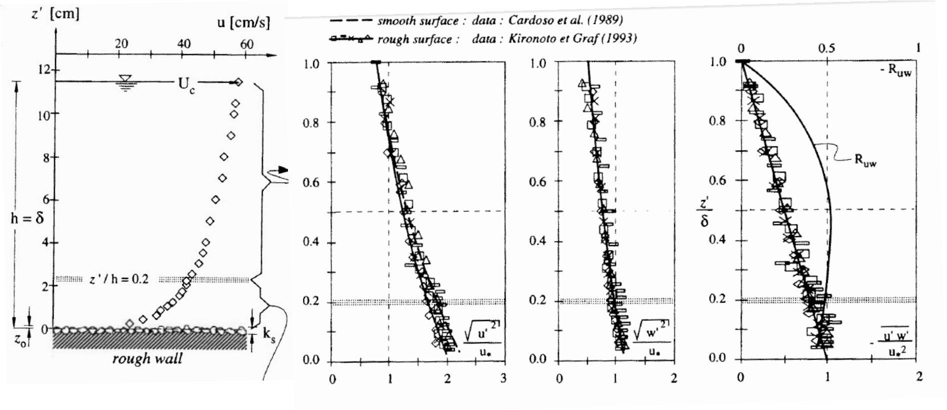
or

$$U = \frac{1}{h}\int_0^h \vec{u}(x,z).\vec{x}\,dh$$

Looking back at the measurements this assumption is reasonable and anyway it is the only way to achieve calculations by hand.

The aspect ratio $\frac{B}{h}$ has an influence on the velocity profile due to the thickness of the lateral boundary layers. For $\frac{B}{h} > 5$ the flow can be considered as 2D.

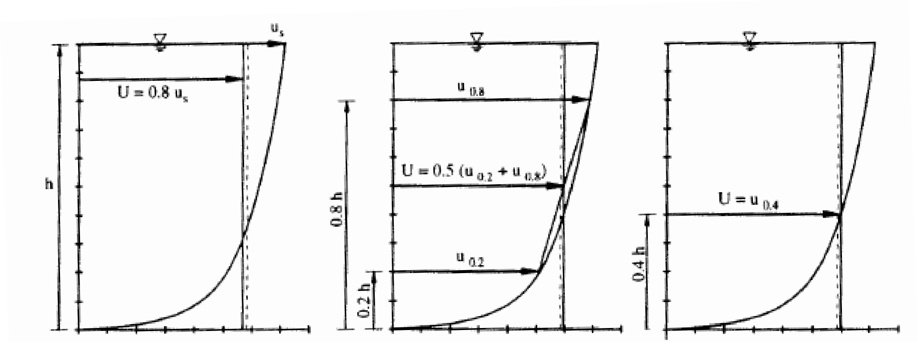
Velocity profile



The profiles presented here corresponds to what you actually measure in an open channel flow under uniform flow conditions.

- The velocity vanishes at the bed and exhibits a logarithmic profile for $z/h < 0.2$
- The Reynolds stress exists ! $< u'^2 >$ and $< w'^2 >$ are the diagonal terms and $< u'w' >$ is the off-diagonal term (=shear stress).

Velocity and Reynolds stress profiles

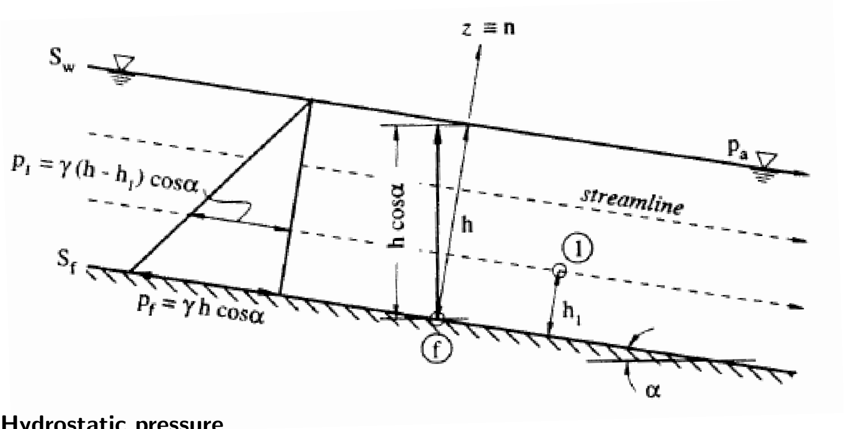


The following formula can be used to estimate the average velocity in a given cross section:

- Prony formula: $U = (0.8 - 0.9)U_{surface}$
- USGS formula : $U = 0.5\ \left(U_{0.2} + U_{0.8}\right)$
- simple formula : $U = U_{0.4}$

where U_α represents the value of $U(z)$ at the elevation $z/h = \alpha$.

Pressure distribution in uniform flows

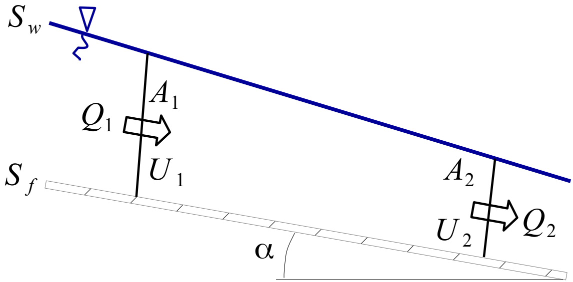


Hydrostatic pressure

Navier-Stokes equation on $(0z)$: $0 = -\dfrac{\partial p}{\partial z} + \rho\ g\cos\alpha$

$\Rightarrow p = \rho\ g\ (h - z)\cos\alpha \simeq \rho\ g\ (h - z)$ for small α $(\alpha < 6^\circ)$

Continuity equation (steady state)



Consider a 2D flow for simplicity: $\overrightarrow{u} = (u,0,w)$

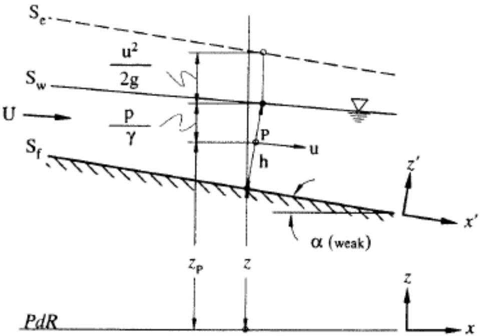
The conservation of mass for an incompressible fluid reads: $\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0$

Integrating over the water depth: $\displaystyle\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) dz = 0$ (w=0 at $z = 0; h$)

$$\text{so that } \frac{\partial}{\partial x}\left(\int_0^h u \; dz \right) = \frac{\partial U \; h}{\partial x} = 0.$$

$$\boxed{Q_1 = Q_2 = Q} \text{ or } U_1 \; A_1 = U_2 \; A_2$$

Total Energy



Total head at elevation z:

$$\frac{p_t(z_p)}{\rho\ g} = \underbrace{\frac{p(z_p)}{\rho\ g}}_{\text{Pressure}} + \underbrace{z_p}_{\text{Potential}} + \underbrace{\frac{u(z_p)^2}{2\ g}}_{\text{Velocity}}$$

expressed in meter of water column.

Total head in a cross section:

$$\boxed{H = \frac{\alpha U^2}{2\ g} + h + z_f}$$

$\alpha = \dfrac{1}{h\ U^3} \displaystyle\int_0^h u^3 \mathrm{d}z$: Energy coefficient

in general, for a turbulent flow $\alpha \approx 1$

Piezometric or reduced pressure: $p^* = p + \rho\ g\ z$

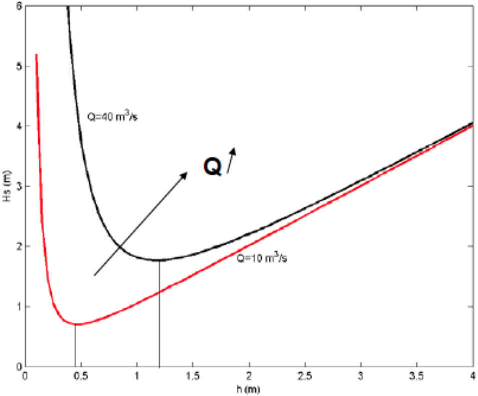
with hydrostatic pressure distribution $p = \rho\ g\ (h - z) \Rightarrow p^* = \rho\ g\ (h - z) + \rho\ g\ z$

$\Rightarrow p^* = \rho\ g\ h = \text{constant}$

Specific energy (1/2)

This concept has been first introduced by Bakmeteff in 1932 and corresponds to the total energy in a cross section with reference to the bed elevation:

$$H = \underbrace{\frac{\alpha U^2}{2\ g} + h}_{H_s:\ \text{Specific Energy}} + z_f \qquad \Rightarrow \qquad H_s = \frac{\alpha U^2}{2\ g} + h$$



Using continuity equation: $Q = U\ A$

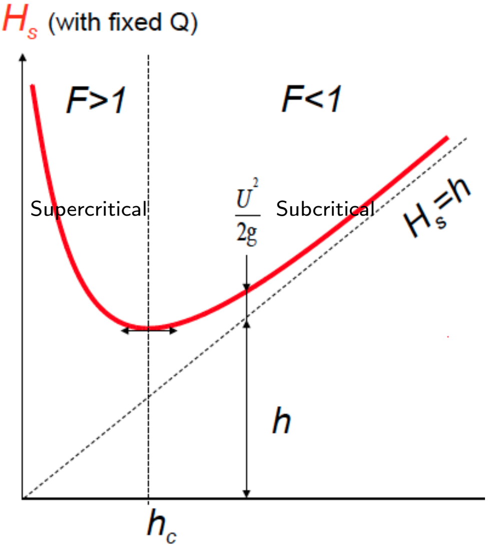
$$\boxed{H_s = \frac{\alpha Q^2}{2\ g\ A^2} + h}$$

For a rectangular cross section channel: $A = b\ h$:

$$H_s = \frac{\alpha Q^2}{2\ g\ b^2\ h^2} + h$$

curve example for $b = 10$ m

Specific energy (2/2)



The minimum of H_s versus h corresponds to the critical depth h_c:

$$H_s^{min} \Longleftrightarrow \left.\frac{\mathrm{d}H_s}{\mathrm{d}h}\right|_{h=h_c} = 0$$

$$\begin{aligned}
\left.\frac{\mathrm{d}H_s}{\mathrm{d}h}\right|_{h=h_c} &= \frac{\alpha Q^2}{2\ g\ b^2}\frac{\mathrm{d}h^{-2}}{\mathrm{d}h} + 1 \\
&= \frac{\alpha Q^2}{2\ g\ b^2}\frac{-2}{h_c^3} + 1 \\
\left.\frac{\mathrm{d}H_s}{\mathrm{d}h}\right|_{h=h_c} &= \frac{-\alpha Q^2}{g\ b^2\ h_c^3} + 1
\end{aligned}$$

$$H_s^{min} \Longleftrightarrow \frac{\alpha Q^2}{g\ b^2\ h_c^3} = F^2 = 1$$

Critical depth: $h_c = \left(\frac{\alpha Q^2}{g\ b^2}\right)^{1/3}$

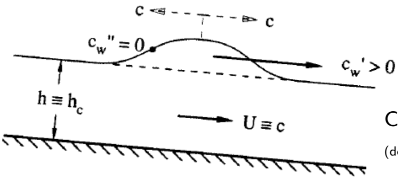
rectangular channel

Waves propagation and flow regimes

Long waves propagates with **celerity**: $\boxed{C^2 = g\ h}$ (rectangular channel in quiescent water).

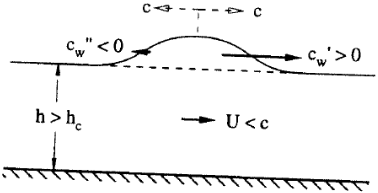
long wave *i.e.* $\dfrac{L}{h} >> 1$

If there is a flow in the channel, the **absolute wave celerity** is given by: $\boxed{C_w = U \pm C}$

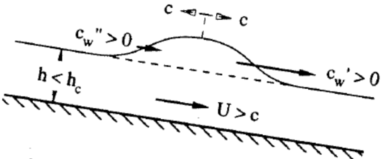


Critical flow: single wave

(downstream)



Subcritical flow: two waves (up- and downstream)



Supercritical flow: two waves (downstream only)

From Navier-Stokes to Shallow water equations

Streamwise component of the Navier-Stokes equation (2D flow $v = 0$)

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{unst.}} + \underbrace{u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}}_{\text{advection}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\rho\ g\ \sin\alpha}_{\text{gravity}} + \underbrace{\frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}}_{\text{total st.}}$$

Assuming:

1. steady flows *i.e.* $\dfrac{\partial}{\partial t} = 0$
2. negligible normal stresses $\tau_{xx} \approx 0; \tau_{zz} \approx 0$
3. weak slopes: $\sin\alpha \approx \tan\alpha \approx S_f$

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g\ S_f + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}$$

Using the hydrostatic pressure distribution $p = \rho\ g\ (h - z)$:

$$u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} + g\ S_f + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}$$

From Navier-Stokes to Shallow water equations

Integrating along the vertical direction (Oz):

$$\int_0^h \underbrace{u\frac{\partial u}{\partial x}}_{=\frac{1}{2}\frac{\partial u^2}{\partial x}} \mathrm{d}z = -\int_0^h g\frac{\partial h}{\partial x}\mathrm{d}z + \int_0^h g\ S_f \mathrm{d}z + \int_0^h \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}\mathrm{d}z$$

for a non-uniform velocity profile: $\int_0^h u^2 \mathrm{d}z = \beta\ h\ U^2$ ($\beta = \mathrm{O}(1)$: Boussinesq coefficient)

$$\frac{1}{2}\frac{\partial \beta\ h\ U^2}{\partial x} = -g\ h\ \frac{\partial h}{\partial x} + g\ h\ S_f + \frac{\tau_{xz}^s - \tau_{xz}^b}{\rho}$$

with

- τ_{xz}^s: shear stress acting at the free surface (by the wind for example ; will be neglected)
- τ_{xz}^b: shear stress acting at the bed

$$\boxed{\underbrace{\frac{1}{2\ h}\frac{\partial \beta\ h\ U^2}{\partial x}}_{inertia} = \underbrace{-g\ \frac{\partial h}{\partial x}}_{pressure} + \underbrace{g\ S_f}_{gravity} - \underbrace{\frac{\tau_{xz}^b}{\rho\ h}}_{bed\ stress}}$$

Depth averaged momentum balance for a steady non-uniform gradually varied flow

Exercices

exercice 1 Compute and estimate the average velocity

The table below gives the values of a vertical velocity profile corresponding to a flow having a water depth of $h = 9.9$ m in a prismatic channel.

z (m)	0.4	0.5	0.8	0.9	1.2	1.3	2.2	3.5	4	4.1	5.3	6	7.3	8	9
u (m/s)	0.47	0.84	1.22	1.34	1.38	1.47	1.63	1.92	1.85	1.86	1.99	1.98	2.08	2.04	1.9

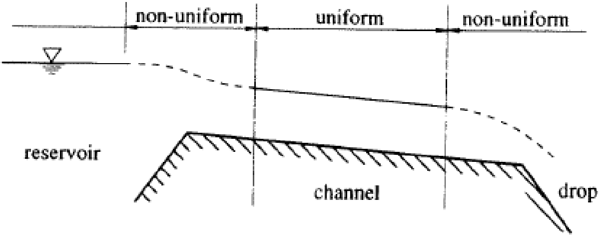
1. Give a method to compute the average velocity U
2. Estimate the average velocity U and compare it with the method proposed in this lecture.

exercice 2 Specific Energy curve for a rectangular cross section channel of width $B = 5$ m

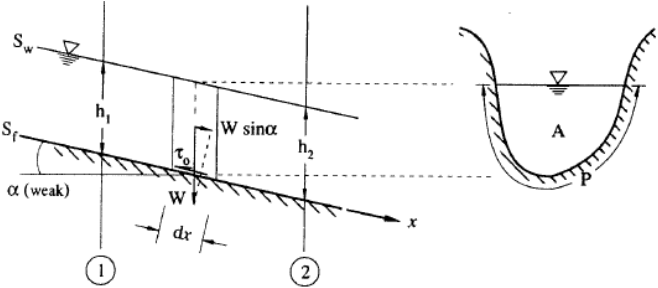
1. Give the relationship for the critical depth h_c as a function of B and Q.
2. Compute the value of h_c for a discharge $Q = 40$ m^3/s.
3. Find the expression of H_c and give its value for $Q = 40$ m^3/s.

Uniform Flows

- The flow in a channel is considered as uniform and steady, if the flow remains constant in the flow direction and in time.
- This configuration is taken as the base configuration for all the other ones.
- In reality uniform flows are rarely encountered. It is only possible in very long prismatic channels far from upstream and downstream boundaries.



Momentum balance



Rectangular channel (from momentum equation):

$$\underbrace{\cancel{\frac{1}{2\,h}\frac{\partial \beta\; h\, U^2}{\partial x}}}_{inertia} = \underbrace{\cancel{-g\frac{\partial h}{\partial x}}}_{pressure} + \underbrace{g\; S_f}_{gravity} - \underbrace{\frac{\tau^b_{xz}}{\rho\; h}}_{bed\;stress} \Longrightarrow \boxed{\underbrace{g\; S_f}_{gravity} = \underbrace{\frac{\tau^b_{xz}}{\rho\; h}}_{bed\;stress}}$$

General case

Friction force: $F_F = \tau_0\; P\; dx$

Gravity force: $F_G = \rho\; g\; A\; \sin\alpha\; dx$

Gravity force = Friction force

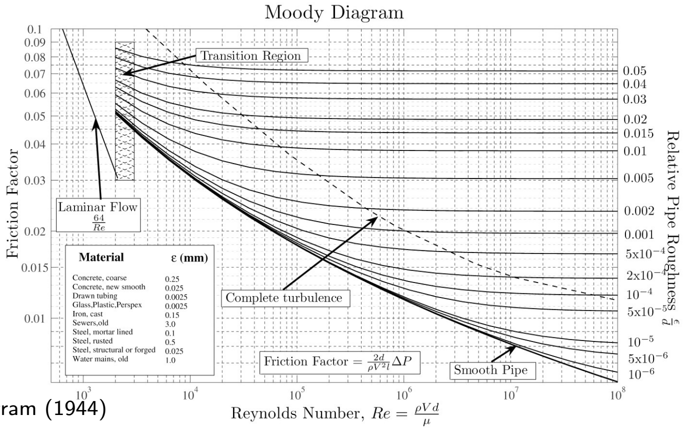
$$\rho\; g\; A\; \sin\alpha\; \cancel{dx} = \tau_0\; P\; \cancel{dx}$$

$$\boxed{\tau_0 = \rho\; g\; \frac{A}{P}\; \sin\alpha = \rho\; g\; R_h\; S_f} \quad (*)$$

Friction coefficients: Darcy-Weisbach equation

The wall shear stress can be modeled according to: $\boxed{\tau_0 = \frac{f}{8}\ \rho\ U^2}$ (**)

where $f = f\left(\frac{k_s}{R_h}, Re\right)$ is the Darcy-Weisbach friction coefficient that depends on the Reynolds number Re and the relative roughness $\frac{k_s}{R_h}$.



Moody diagram (1944)

Friction coefficients: Chézy formula

Identifying (*) and (**) gives: $\frac{f}{8}\ \rho\ U^2 = \rho\ g\ R_h\ S_f$

that can be recasted into: $U = \sqrt{\frac{8\ g}{f}}\ R_h^{1/2}\ S_f^{1/2}$ or $\boxed{U = C\ R_h^{1/2}\ S_f^{1/2}}$

which is known as the Chézy equation where $C = \sqrt{8\ g/f}$ is the **Chézy coefficient ($m^{1/2}/s$)**. Chézy was a French civil and hydraulic engineer who establishes this first law for uniform flows in 1775 (one year before US declaration of independence).

It is usually more convenient to use the discharge in place of the average velocity:

$$Q = U\ A$$

$$\boxed{Q = C\ A\ R_h^{1/2}\ S_f^{1/2}}$$

Table of typical Chézy coefficient values

Description of Channel	Chezy Coefficient
Many grove heights of flood waters	7 - 12.5
Many weeds as high as water	12.5 - 20
Base of channel is clean with a little to moderate grove on the cliff wall channel	20 - 30
Channel with a bit of short grassy weeds	30 - 45
Channel is clean and not a new channel, it has been decaying	40 - 55

Friction coefficients: Manning-Strickler

The major limitation of the Chézy formula resides in the lack of dependency of C to the relative roughness k_s/R_h.

Manning (1881), Irish engineer, and Strickler (1920-1925), Autrichian engineer, independently proposed the following parametrization for C:

$$C = \frac{R_h^{1/6}}{n} = K_s \; R_h^{1/6}$$

n: Manning coefficient $(s/m^{1/3})$
K_s: Strickler coefficient $(m^{1/3}/s)$

Introducing this parametrization into the Chézy equation

$$U = C \; R_h^{1/2} \; S_f^{1/2}$$

leads to the Manning-Strickler equation:

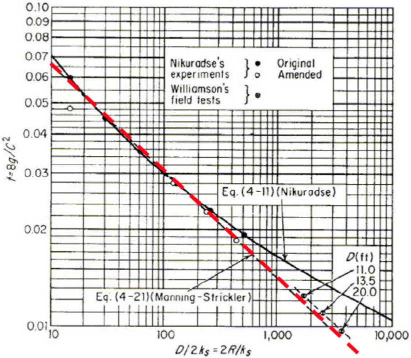
$$\boxed{U = K_s \; R_h^{2/3} \; S_f^{1/2} = \frac{1}{n} R_h^{2/3} \; S_f^{1/2}}$$

or

$$\boxed{Q = K_s \; A \; R_h^{2/3} \; S_f^{1/2} = \frac{1}{n} \; A \; R_h^{2/3} \; S_f^{1/2}}$$

This parametrization made the unanimity.

Henderson (1966)

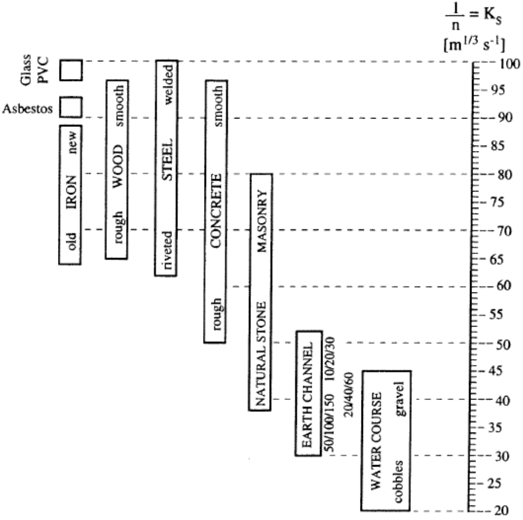


Friction coefficients: Manning-Strickler

The values of K_s or n has been tabulated by many authors, they mostly depends on the nature of the channel's bed and banks (roughness). The choice of the values remains somehow subjectives and **needs to be calibrated using measurements**.

The other problem is that these friction **coefficients are dimensional**. However, the values are the same in the metric and Imperial/US systems. The correction is made on the definition of the Chézy coefficient for the **US system**:

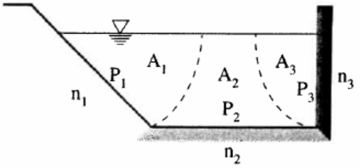
$$C = \frac{1.48\ R_h^{1/6}}{n} = 1.48\ K_s\ R_h^{1/6}$$



See HECRAS 4.1 Hydraulic reference table 3-1 p.80 (from Chow, 1959)

Friction coefficients: Composite roughness

In case the roughness is not homogeneous over the entire cross section it becomes necessary to compute an equivalent friction coefficient.



Following Einstein (see chow, 1959) the cross section A can be divided into N parts of almost homogeneous roughness n_i having a wetted area A_i and wetted perimeter P_i. Assuming that the average velocity is identical in all the sub cross sections:

$$U_1 = U_2 = ... = U_N = U$$

one can write:

$$U = \frac{1}{n_e} \left(\frac{A}{P}\right)^{2/3} S_f^{1/2} = \frac{1}{n_1} \left(\frac{A_1}{P_1}\right)^{2/3} S_f^{1/2} = ... = \frac{1}{n_N} \left(\frac{A_N}{P_N}\right)^{2/3} S_f^{1/2}$$

using the equality $A = \sum_i A_i$ leads to:

$$\boxed{n_e = \left(\frac{n_i^{3/2} \ P_i}{P}\right)^{2/3}}$$

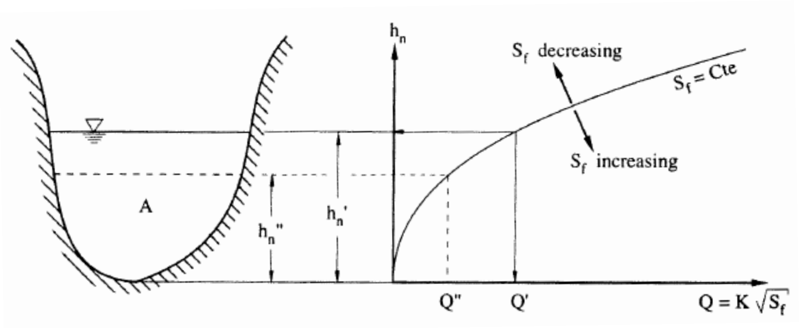
Conveyance

To determine the discharge Q in a channel one needs to know the channel's geometry (A, P), roughness (n) and longitudinal slope (S_f).

$$Q = \frac{1}{n} A \ R_h^{2/3} \ S_f^{1/2} = K(h) \ S_f^{1/2}$$

where $K(h)$ is the channel conveyance that depends only on the water depth h. This depth is usually called the **normal depth** and is denoted as h_n.

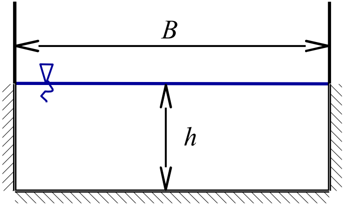
The conveyance characterizes the water transport capacity of a given cross section.



Normal depth for a rectangular channel

Wetted area: $A = B\ h$

Wetted perimeter: $P = B + 2\ h$



Manning-Strickler: $Q = \frac{1}{n} A\ R_h^{2/3}\ S_f^{1/2}$

with hydraulic radius: $R_h = \frac{A}{P}$

Rectangular cross section:

$$R_h = \frac{B\ h}{B + 2\ h}$$

$$Q = \frac{1}{n} \frac{(B\ h)^{5/3}}{(B + 2\ h)^{2/3}}\ S_f^{1/2}$$

Normal depth:

$$\boxed{h_n = \left(\frac{n\ Q}{B\ S_f^{1/2}} \right)^{3/5} \left(1 + \frac{2\ h}{B} \right)^{2/5}}$$

Wide rectangular channel: $B >> h \rightarrow B + 2\ h \approx B$

$$R_h = \frac{B\ h}{B + 2\ h} \approx h$$

$$Q = \frac{1}{n} B\ h^{5/3}\ S_f^{1/2}$$

Normal depth: $\boxed{h_n^{WC} = \left(\frac{n\ Q}{B\ S_f^{1/2}} \right)^{3/5}}$

Wide Channel assumption (WC)

Application to the Delaware river

Geometric data:

Length: $L = 484$ km

Average slope: $S_f = 0.00076$

Discharges at Trenton:

$$Q_{min} = 122 \text{ m}^3/\text{s}$$
$$Q_{ave} = 371 \text{ m}^3/\text{s}$$
$$Q_{max} = 9{,}316 \text{ m}^3/\text{s}$$

Width at Trenton: $B \approx 300$ m

Manning coefficient : $n = 0.035$ s/m$^{1/3}$

Compute the critical and normal depths at Trenton for the three given disharge values.

For the computations, assume a rectangular cross section for the river.

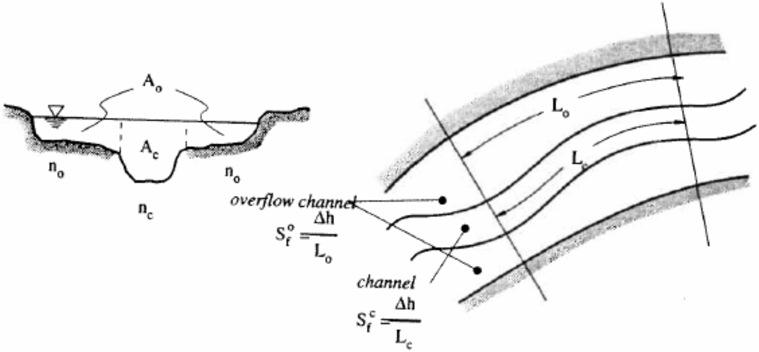
source: Wikipedia

Composite cross section

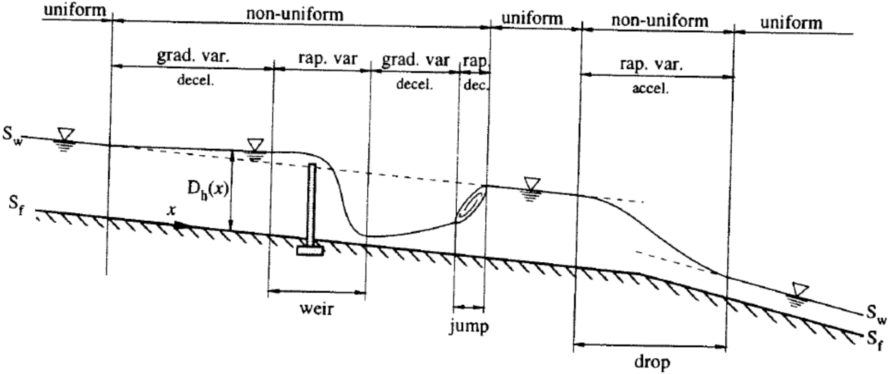
During a flood, the flow takes place in two very different part of the channel: the main channel and the floodplain (overbanks). The roughness and the slope of these two parts can be very different and this situation can be dealt with the following formula:

$$Q = Q_c + Q_o = \frac{1}{n_c} A_c \ R_{h_c}^{2/3} \ \frac{\Delta h}{L_c}^{1/2} + \frac{1}{n_o} A_o \ R_{h_o}^{2/3} \ \frac{\Delta h}{L_o}^{1/2}$$

where P_c and P_o should only include fluid-bed or banks contact lines *i.e* no fluid-fluid contact lines.

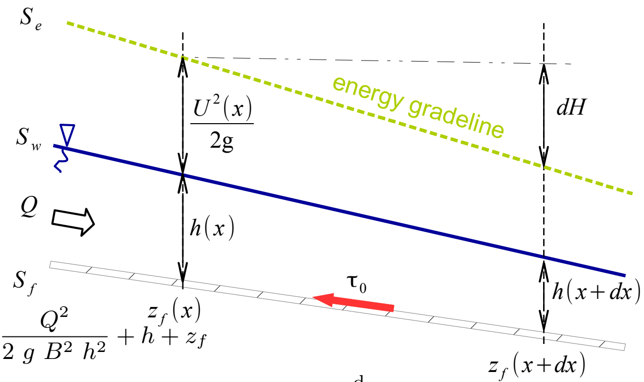


Non-Uniform Flows: Introduction



Non uniform flows are encountered when the water depth varies along the streamwise direction. Two different cases will be considered, if the **free surface curvature is small** compared with the channel slope the flow is called **Gradually Varied Flow (GVF)** and if the **free surface curvature is important** the flow will be called **Rapidly Varied Flow (RVF)**. The hypotheses and equations that govern these two non-uniform flow regimes are different and will be detailed hereafter.

Water depth equation



Channel

- prismatic: $\frac{\partial B; S_f}{\partial x} = 0$
- rectangular of width B

Energy equation:

$$H = \frac{U^2}{2\ g} + h + z_f = \frac{Q^2}{2\ g\ B^2\ h^2} + h + z_f$$

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q^2}{2\ g\ B^2\ h^2}\right) + \frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}z_f}{\mathrm{d}x}$$

derivation $\frac{\mathrm{d}}{\mathrm{d}x}$

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{Q^2}{g\ B^2\ h^3}\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}z_f}{\mathrm{d}x}$$

$\frac{\mathrm{d}h^{-2}}{\mathrm{d}x} = -\frac{2}{h^3}\frac{\mathrm{d}h}{\mathrm{d}x}$ and $Fr^2 = \frac{Q^2}{g\ B^2\ h^3}$

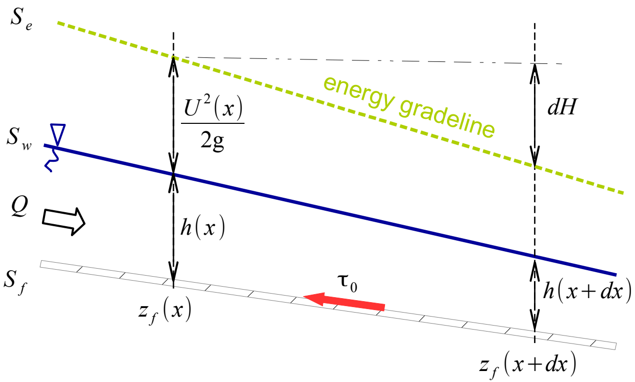
$$-S_e = \left(1 - Fr^2\right)\frac{\mathrm{d}h}{\mathrm{d}x} - S_f$$

$\frac{\mathrm{d}H}{\mathrm{d}x} = -S_e$ and $\frac{\mathrm{d}z_f}{\mathrm{d}x} = -S_f$

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_f - S_e}{1 - Fr^2}}$$

This equation remains true for an arbitrary cross section

Water depth equation



Water depth equation:

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_f - S_e}{1 - Fr^2}}$$

Energy slope: $S_e = -\frac{\mathrm{d}H}{\mathrm{d}x}$

$\rightarrow$ corresponds to the slope of the energy grade line = energy dissipated by friction

$\rightarrow$ for a uniform flow: $S_e = S_f$ and $Q = \frac{1}{n} A \ R_h^{2/3} \ S_f^{1/2}$ (assuming a Manning-Strickler law)

$\rightarrow$ for a non uniform GVF: $S_e \neq S_f$ and $Q = \frac{1}{n} A \ R_h^{2/3} \ S_e^{1/2} \Rightarrow \boxed{S_e = \left(\frac{n \ Q}{A \ R_h^{2/3}}\right)^2}$

S_e is the slope that an equivalent uniform flow would have in a channel having a bed slope equal to the energy gradeline one.

Water depth equation

prismatic rectangular channel and Manning-Strickler law

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_f - S_e}{1 - Fr^2} \Longrightarrow \frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - S_e/S_f}{1 - Fr^2}$$

with $S_e = \left(\frac{n\ Q}{B\ h^{5/3}}\right)^2$ we can write $\frac{S_e}{S_f} = \left(\frac{n\ Q}{B\ S_f^{1/2}\ h^{5/3}}\right)^2 = \left(\frac{h_n}{h}\right)^{10/3}$

and $Fr^2 = \frac{Q^2}{g\ B^2\ h^3} = \left(\frac{h_c}{h}\right)^3$

The water depth equation can be written as: $\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \left(\frac{h_n}{h}\right)^{10/3}}{1 - \left(\frac{h_c}{h}\right)^3}}$

This equation is valid only for a wide rectangular channel assuming a Manning-Strickler law

Remarks:

1. If $h = h_n$ then $\frac{\mathrm{d}h}{\mathrm{d}x} = 0$ and h remains constant: the uniform flow is stable.
2. If $h = h_c$ then $\frac{\mathrm{d}h}{\mathrm{d}x} \longrightarrow \infty$ and the free surface is vertical ! Incompatible with slow variation in the streamwise direction.

Water depth variation

prismatic rectangular channel and Manning-Strickler law

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \left(\frac{h_n}{h}\right)^{10/3}}{1 - \left(\frac{h_c}{h}\right)^3}}$$

(wide rectangular channel assuming a Manning-Strickler law)

Subcritical flow: $h_n > h_c$ 3 cases

1. $\boxed{h > h_n > h_c}: \Longrightarrow \boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{>0}{>0} > 0}$

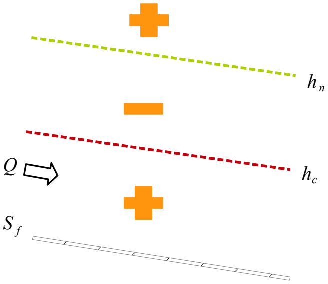
$$\frac{h_n}{h} < 1 \text{ and } \frac{h_c}{h} < 1 \Longrightarrow 1 - \left(\frac{h_n}{h}\right)^{10/3} > 0 \text{ and } 1 - \left(\frac{h_c}{h}\right)^3 > 0$$

2. $\boxed{h_n > h > h_c}: \Longrightarrow \boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{<0}{>0} < 0}$

$$\frac{h_n}{h} > 1 \text{ and } \frac{h_c}{h} < 1 \Longrightarrow 1 - \left(\frac{h_n}{h}\right)^{10/3} < 0 \text{ and } 1 - \left(\frac{h_c}{h}\right)^3 > 0$$

3. $\boxed{h_n > h_c > h}: \Longrightarrow \boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{<0}{<0} > 0}$

$$\frac{h_n}{h} > 1 \text{ and } \frac{h_c}{h} > 1 \Longrightarrow 1 - \left(\frac{h_n}{h}\right)^{10/3} < 0 \text{ and } 1 - \left(\frac{h_c}{h}\right)^3 < 0$$



Water depth variation

prismatic rectangular channel and Manning-Strickler law

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \left(\dfrac{h_n}{h}\right)^{10/3}}{1 - \left(\dfrac{h_c}{h}\right)^{3}}}$$

(wide rectangular channel assuming a Manning-Strickler law)

Supercritical flow: $h_c > h_n$ 3 cases

1. $\boxed{h > h_c > h_n}$: $\Longrightarrow$ $\boxed{\dfrac{\mathrm{d}h}{\mathrm{d}x} = \dfrac{>0}{>0} > 0}$

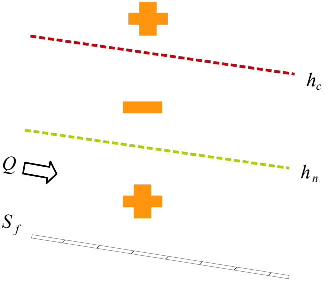
   $\dfrac{h_n}{h} < 1$ and $\dfrac{h_c}{h} < 1 \Longrightarrow 1 - \left(\dfrac{h_n}{h}\right)^{10/3} > 0$ and $1 - \left(\dfrac{h_c}{h}\right)^{3} > 0$

2. $\boxed{h_c > h > h_n}$: $\Longrightarrow$ $\boxed{\dfrac{\mathrm{d}h}{\mathrm{d}x} = \dfrac{>0}{<0} < 0}$

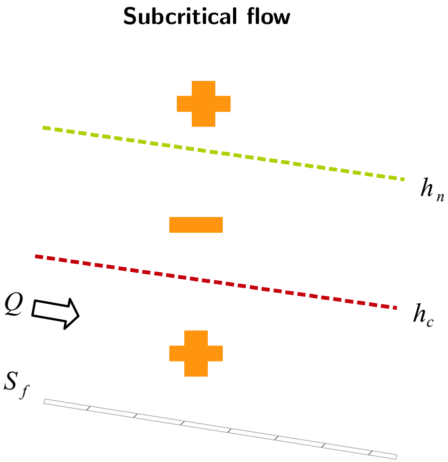
   $\dfrac{h_n}{h} < 1$ and $\dfrac{h_c}{h} > 1 \Longrightarrow 1 - \left(\dfrac{h_n}{h}\right)^{10/3} > 0$ and $1 - \left(\dfrac{h_c}{h}\right)^{3} < 0$

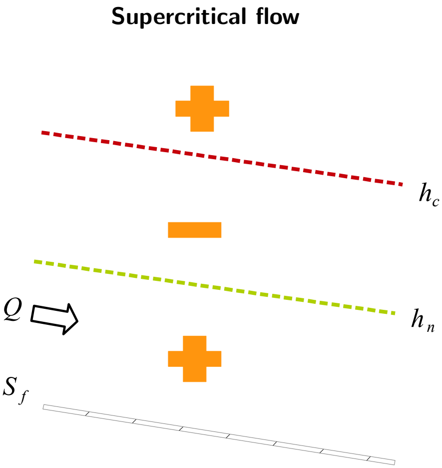
3. $\boxed{h_c > h_n > h}$: $\Longrightarrow$ $\boxed{\dfrac{\mathrm{d}h}{\mathrm{d}x} = \dfrac{<0}{<0} > 0}$

   $\dfrac{h_n}{h} > 1$ and $\dfrac{h_c}{h} > 1 \Longrightarrow 1 - \left(\dfrac{h_n}{h}\right)^{10/3} < 0$ and $1 - \left(\dfrac{h_c}{h}\right)^{3} < 0$



Water depth variation: synthesis





Water depth equation

For a prismatic **wide rectangular channel and Chézy law**, the water depth equation is:

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \left(\dfrac{h_n}{h}\right)^3}{1 - \left(\dfrac{h_c}{h}\right)^3}}$$

In the **general case**, the water depth equation is:

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \dfrac{(Q/A)^2}{C^2\ R_h\ S_f}}{1 - \left(\dfrac{Q^2\ B}{g\ A^3}\right)^2}}$$

Chézy law

or

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \dfrac{n^2 (Q/A)^2}{R_h^{4/3}\ S_f}}{1 - \left(\dfrac{Q^2\ B}{g\ A^3}\right)^2}}$$

Manning law

The notion of critical slope

If the uniform flow is in the critical regime *i.e:* $h_n = h_c$

Then the channel is at the **critical slope** S_c

Demo:

$$Q = \frac{1}{n} A\ R_h^{2/3}\ S_c^{1/2} \qquad \frac{Q^2\ B}{g\ A^3} = 1$$

$$Q^2 = \frac{1}{n^2} A^2\ R_h^{4/3}\ S_c \qquad Q^2 = \frac{g\ A^3}{B}$$

$$\frac{1}{n^2}\ R_h^{4/3}\ S_c = \frac{g\ A}{B}$$

$$\boxed{S_c = \frac{n^2\ g\ A}{B\ R_h^{4/3}}} \longrightarrow \boxed{S_c = \frac{n^2\ g}{h^{1/3}}}$$

wide rectangular channel and Manning-Strickler law

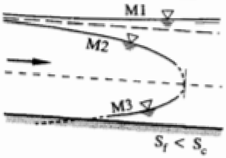
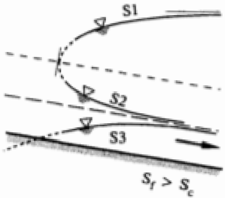
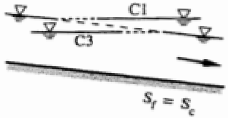
The different shapes of water surface profiles

The classification of water surface profiles is usually done according to the bed slope S_f (Graf and Altinakar, 1998):

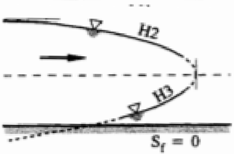
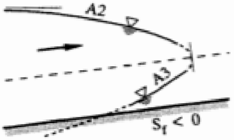
	$S_f < S_c$	channels on Mild slope	:	M
$\boxed{S_f > 0}$	$S_f > S_c$	channels on Steep slope	:	S
	$S_f = S_c$	channels on Critical slope	:	C
$\boxed{S_f = 0}$		channels on Horizontal slope	:	H
$\boxed{S_f < 0}$		channels on Adverse slope	:	A

Each curve is composed of different branches depending on the actual value of the water depth compared with the normal and critical depths.

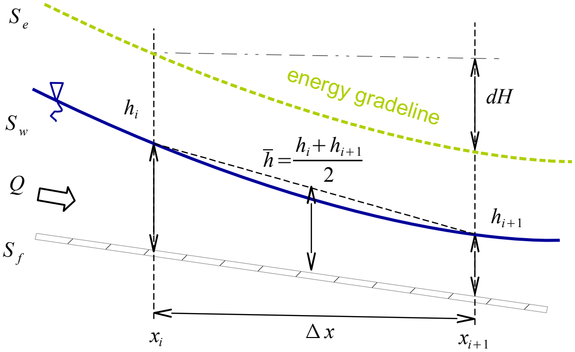
The different shapes of water surface profiles

Conditions Eq. 4.8a	$\frac{h_n}{h}$	Sign num.	$\frac{h_c}{h}$	Sign den.	Sign dh/dx	Change of flow depth	Name	Profiles *vertical scale exaggerated*
$S_f > 0$								
$S_f < S_c$	< 1	+	< 1	+	+	increase	M1	
$h_n > h_c$	< 1	+	> 1	-		not possible		
	> 1	-	< 1	+	-	decrease	M2	
	> 1	-	> 1	-	+	increase	M3	
$S_f > 0$								
$S_f > S_c$	< 1	+	< 1	+	+	increase	S1	
$h_n < h_c$	< 1	+	> 1	-	-	decrease	S2	
	> 1	-	> 1	-	+	increase	S3	
$S_f > 0$								
$S_f = S_c$	< 1	+	< 1	+	+	increase	C1	
$h_n = h_c$	> 1	-	> 1	-	+	increase	C3	

The different shapes of water surface profiles

Conditions Eq. 4.8a	$\frac{h_n}{h}$	Sign num.	$\frac{h_c}{h}$	Sign den.	Sign dh/dx	Change of flow depth	Name	Profiles *vertical scale exaggerated*
$S_f = 0$ $h_n = \infty$		-	< 1	+	-	decrease	H2	
		-	> 1	-	+	increase	H3	
$S_f < 0$ $h_n < 0$	< 1	-	< 1	+	-	decrease	A2	
	< 1	-	> 1	-	+	increase	A3	

Methods for computing water surface profiles



For a prismatic wide rectangular channel: $\dfrac{\mathrm{d}h}{\mathrm{d}x} = S_f \dfrac{1-(h_n/h)^{10/3}}{1-(h_c/h)^3}$

which can also be written as: $\mathrm{d}h = S_f \dfrac{1-(h_n/h)^{10/3}}{1-(h_c/h)^3}\ \mathrm{d}x$

integrating between x_i and x_{i+1} gives: $\boxed{h_{i+1} - h_i = S_f \dfrac{1-\left(h_n/\bar{h}\right)^{10/3}}{1-\left(h_c/\bar{h}\right)^3}\ (x_{i+1} - x_i)}$

Methods for computing water surface profiles

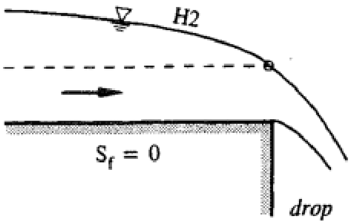
The discrete equation: $h_{i+1} - h_i = S_f \dfrac{1 - \left(h_n/\bar{h}\right)^{10/3}}{1 - \left(h_c/\bar{h}\right)^3} \, (x_{i+1} - x_i)$

can be used to compute the water surface profile in the following way:

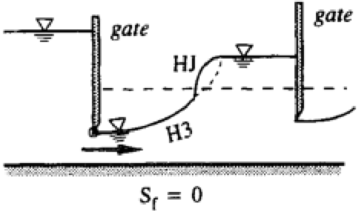
1. For a reach of a small distance Δx one can compute the water depth variation Δh; this method is know as the standard step method.
2. For a small water depth difference Δh one can compute the distance Δx between these two water depths; this method is know as the depth variation or direct method.
3. Before starting the computation one has to establish the control sections where a known relationship between the water depth and the discharge exists (could be a an outlet/inlet of a channel, a weir, a gate or a hydraulic drop).
4. Computations proceed upstream for subcritical flows ($Fr < 1$) and downstream for supercritical flows ($Fr > 1$).
5. The standard step method is more time consuming but usually more precise. This is the method used in HEC-RAS.

Flow profiles examples: horizontal channel

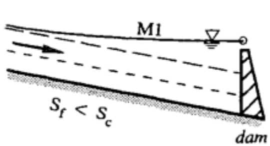
Free overfall or drop: no longer a gradually varied flow but in a subcritical regime the free overfall controls the flow depth upstream.

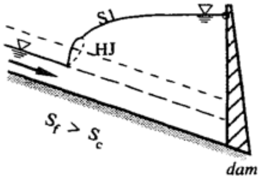


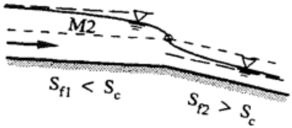
If the sluice gate forces the flow below the critical depth a supercritical regime is observed downstream the gate so the water surface profile has to be computed in the downstream direction. The downstream gate imposes a water level above the critical depth upstream so the flow has to be subcritical there. The water surface profile has to be computed in the upstream direction. The matching of this two branches a hydraulic jumps takes place to ensure the transition from a supercritical to a subcritical flow.

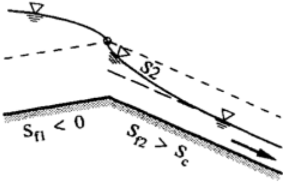


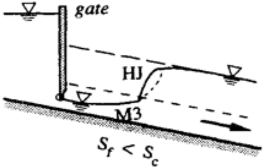
Flow profiles examples

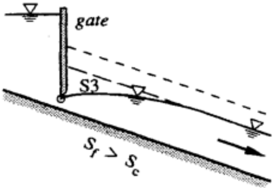






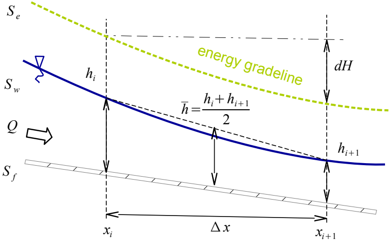






Standard step method (Δx fixed)

Solve: $$h_{i+1} - h_i = \underbrace{S_f \frac{1 - \left(h_n/\bar{h}\right)^{10/3}}{1 - \left(h_c/\bar{h}\right)^3}}_{=\overline{\frac{\mathrm{d}h}{\mathrm{d}x}}} (x_{i+1} - x_i)$$



In this example we assume that the flow is supercritical and so we compute the water depth profile in the downstream direction. The value of Q and h_i at the upstream cross section are given. **The question is: find the value of h_{i+1} at the position x_{i+1}.**

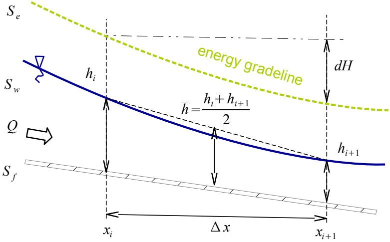
The space step Δx is given by: $\Delta x = x_{i+1} - x_i$. The solution procedure is as follows:

1. Choose a value for h_{i+1}^k (extrapolation, zero gradient, secant method, ...)
2. Compute $\bar{h}^k = \dfrac{h_{i+1}^k + h_i}{2}$ and $\overline{\dfrac{\mathrm{d}h}{\mathrm{d}x}}(\bar{h}^k)$
3. Compute the new h_{i+1}^{k+1} value using: $h_{i+1}^{k+1} = h_i + \overline{\dfrac{\mathrm{d}h}{\mathrm{d}x}}(\bar{h}^k)\ \Delta x$

   (non linear equation $\Rightarrow$ iterative solution)
4. if $\left|h_{i+1}^{k+1} - h_{i+1}^k\right| > \epsilon$ then step 1 to 3 are repeated with a new guess for h_{i+1}

Detph variation method (Δh fixed)

Solve: $x_{i+1} - x_i = \underbrace{S_f^{-1} \frac{1 - \left(h_c/\bar{h}\right)^3}{1 - \left(h_n/\bar{h}\right)^{10/3}}}_{=\overline{\frac{\mathrm{d}x}{\mathrm{d}h}}} (h_{i+1} - h_i)$



Again we assume that the flow is supercritical and the value of Q and h_i at the upstream cross section are given. **The question is: find the value of x_{i+1} at which the water depth is h_{i+1}.**

The depth variation Δh is given by: $\Delta h = h_{i+1} - h_i$. The solution procedure is as follows:

1. Compute $\bar{h}^k = h_i + \Delta h$ and $\overline{\frac{\mathrm{d}x}{\mathrm{d}h}}(\bar{h})$
2. Compute the value of Δx using: $\Delta x = \overline{\frac{\mathrm{d}x}{\mathrm{d}h}}(\bar{h})\ \Delta h$
3. Compute the value of $x_{i+1} = x_i + \Delta x$ then move on to the next step h_{i+2} by using the known value at h_{i+1} at x_{i+1}.

Water surface profile computations: some remarks

1. The two methods presented above can **also be used for an arbitrary cross section** geometry. The equation to be solved is then:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = S_f \frac{1 - \dfrac{n^2 (Q/A)^2}{R_h^{4/3}\ S_f}}{1 - \left(\dfrac{Q^2}{g}\dfrac{B}{A^3}\right)^2}$$

   more complicated but it's not a big deal with a computer.

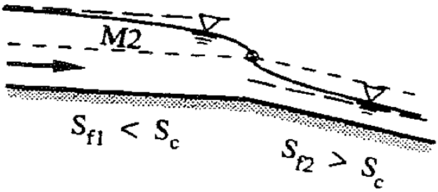
2. The **depth variation method** is the **easiest method** for computation by hand (no iterations)

3. The **standard step method** is **more accurate** and more convenient for numerical computations especially for non prismatic channel. This is the method used in **HECRAS**.

4. **Other methods exist** (see Chow 2009 or Graf and Altinakar 1998 for details), some of them are semi analytic but slightly more complicated to use.

Rapidly Varied Flows: introduction

- Upstream and downstream of the critical depth h_c the flow is rapidly varied.
- The assumptions of the Gradually Varied Flow are no longer valid: slow changes with x, small curvature of the streamlines and hydrostatic pressure distribution.
- The passage of the critical depth is usually associated with a sudden change in water depth like a hydraulic jump when the flow depth increases or a hydraulic drop when the flow depth decreases.
- Rapidly varied flows usually corresponds to control sections of the flow profile.



Weirs and spillways

A weir or a spillway is a device or structure that allows to measure or control the discharge Q in channel. The flow passes over the weir towards the downstream. It is characterized by its height H_D and its length in the cross section L_D.

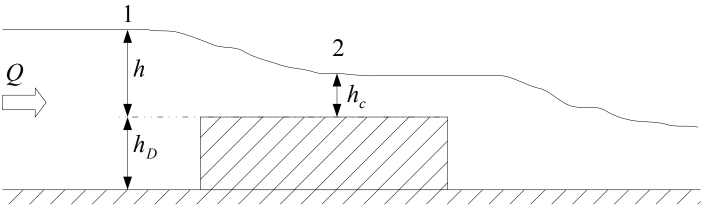
Only unsubmerged weirs will be studied here but remember that a weir can be submerged.

https://en.wikipedia.org/wiki/Radyr

http://www.hydrocad.net/weir1.htm

dobbs weir

Broad crested weir



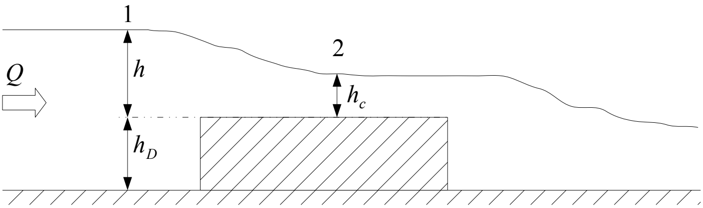
Assumptions:

- The flow is supercritical downstream.
- The flow is subcritical upstream.
- Negligible head loss between the upstream cross section and the weir.
- Horizontal bed

In this condition the total head is conserved:

- Upstream: $H = h_D + h + \cancel{\dfrac{Q^2}{2\ g\ L_D^2\ (h_D+h)^2}}$ (subcritical flow upstream $\Rightarrow$ negligible)
- Critical section: $H = h_D + h_c + \dfrac{Q^2}{2\ g\ L_D^2\ h_c^2} = h_D + \dfrac{3}{2}\left(\dfrac{Q^2}{g\ L_D^2}\right)^{1/3}$

Broad crested weir



$$\cancel{h_D} + h = \cancel{h_D} + \frac{3}{2}\left(\frac{Q^2}{g\ L_D^2}\right)^{1/3} \longrightarrow Q = L_D \left(\frac{2}{3}\right)^{3/2} \sqrt{g}\ h^{3/2} = L_D \frac{2}{3^{3/2}}\ \sqrt{2\ g}\ h^{3/2}$$

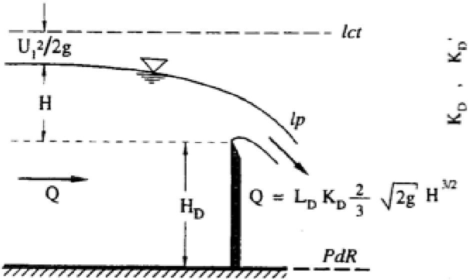
$$\boxed{Q = L_D\ 0.577\ \frac{2}{3}\ \sqrt{2\ g}\ h^{3/2}}$$

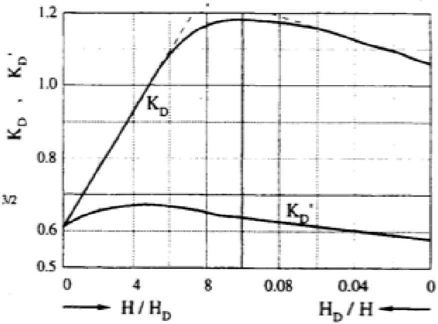
In the general case, the coefficient 0.577 is replaced by a discharge coefficient K_D that depends on the weir geometry:

$$\boxed{Q = L_D\ K_D\ \frac{2}{3}\ \sqrt{2\ g}\ h^{3/2}}$$

Sharp crested weir

Warning: notation change $h \to H$





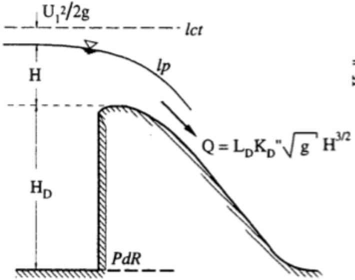
Weir equation: $\boxed{Q = L_D \; K_D \; \frac{2}{3} \; \sqrt{2 \; g} \; H^{3/2}}$ with $K_D \left(\frac{H}{H_D} \right) \in [0.6; 1.2]$

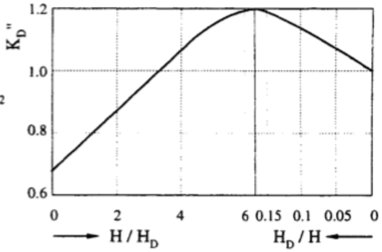
When $\frac{H}{H_D}$ increases the kinetic energy contribution is no more negligible and one should use the following weir equation: $\boxed{Q = L_D \; K_D' \; \frac{2}{3} \; \sqrt{2 \; g} \; \mathcal{H}^{3/2}}$ where $\mathcal{H} = H + \frac{U^2}{2 \; g}$

with $K_D' \left(\frac{H}{H_D} \right) \in [0.58; 0.67]$

Spillways

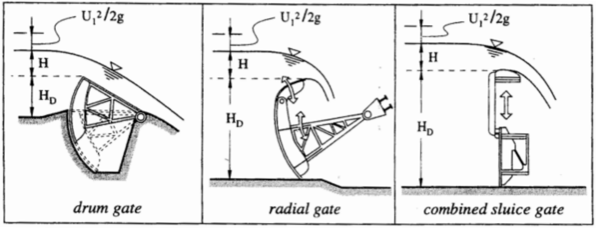
Warning: notation change $h \rightarrow H$



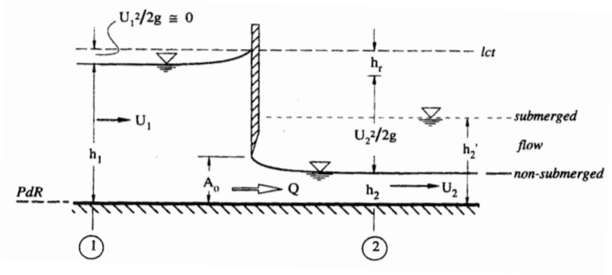


Weir equation: $\boxed{Q = L_D\ K''_D \sqrt{g}\ H^{3/2}}$ with $K_D\left(\dfrac{H}{H_D}\right) \in [0.65; 1.2]$

Mobile spillways



Underflow gates



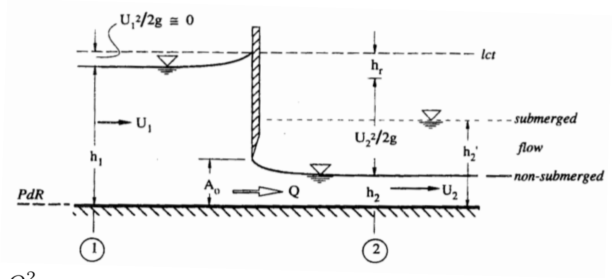
Assumptions:

- The flow is subcritical upstream.
- The flow can be supercritical downstream (non-submerged) or subcritical (submerged).
- Negligible head loss between cross section 1 and 2.
- Horizontal bed

In this condition the specific head is conserved:

- Upstream: $H = h_1 + \cancel{\dfrac{Q^2}{2\ g\ B^2\ h_1^2}}$ (subcritical flow upstream $\Rightarrow$ negligible)
- Downstream: $H = h_2 + \dfrac{Q^2}{2\ g\ B^2\ h_2^2}$

Underflow gates



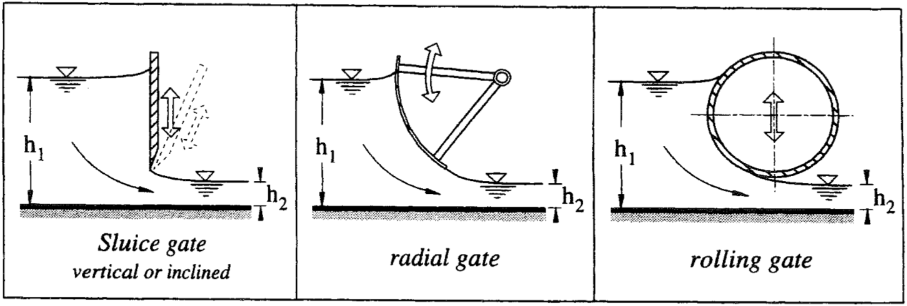
$$h_1 = h_2 + \frac{Q^2}{2\ g\ B^2\ h_2^2} \longrightarrow Q = h_2\ B\ \sqrt{2\ g\,(h_1 - h_2)}$$

In the non-submerged case, $h_2 << h_1$ and the *vena contracta* is parametrized using a contraction coefficient: $C_c = \dfrac{h_2}{h_0}$ with $C_c \in [0.5; 1]$: $\boxed{Q = C_c\ h_0\ B\ \sqrt{2\ g\ h_1}}$

The general gate equation is given by: $\boxed{Q = K_v\ A_0\ \sqrt{2\ g\ h_1}}$

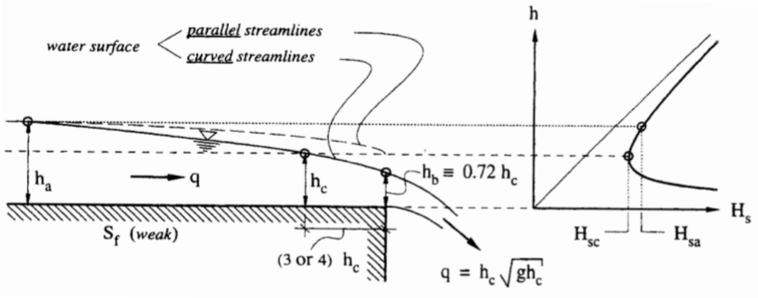
and the gate coefficient $K_v = C_q\ C_c/\sqrt{1 + C_c(A_0/A_1)}$ with $C_q \in [0.95; 0.99]$ and $C_c = A_2/A_0$.

Underflow gates



Many underflow gates geometry exists, the basic equation remains the same the difference is in the K_v coefficient that depends on geometrical parameters (should be given by the manufacturer). Underflow gates are very sensitive devices that needs to be carefully calibrated!

Hydraulic drop and free overfall

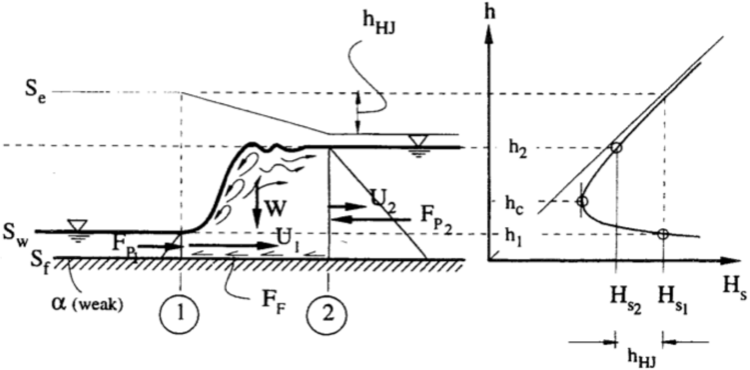


When a subcritical open channel flow freely discharge into the atmosphere, this is a hydraulic drop or free overfall. This situation is the limit case of Mild slope channel flowing in a steep slope channel (with infinite slope). The flow has to experience a transition from subcritical to supercritical so the water depth has to be critical somewhere.

The critical depth is observed slightly before the outlet: $\boxed{h_b = 0.72 - 0.75\ h_c}$

In fact the flow can not be assumed gradually varied anymore and so the pressure is not hydrostatic at the outlet. This can be demonstrated using the momentum balance between h_a and the free fall.

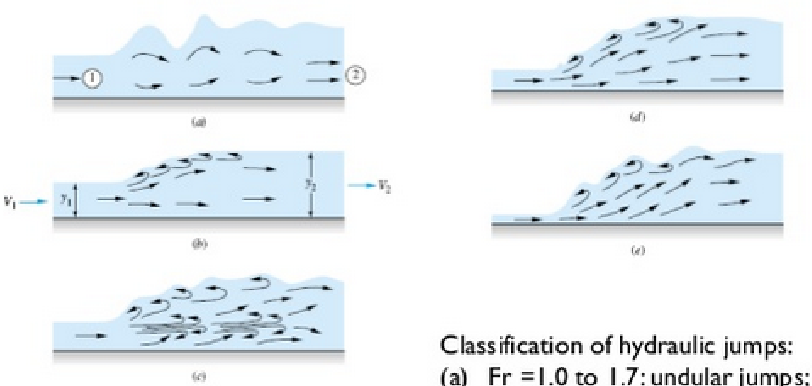
Hydraulic jump



The hydraulic jump is encountered when the flow passes abruptly from supercritical to subcritical in the downstream direction.

- sudden water surface elevation (discontinuity) or stationary wave
- very turbulent motion: undulation and air entrainment
- form of a aerated wave breaking in roller
- causes strong dissipation of energy: h_{HJ}

Hydraulic jump: classification



Classification of hydraulic jumps:

(a) Fr =1.0 to 1.7: undular jumps;

(b) Fr =1.7 to 2.5: weak jump;

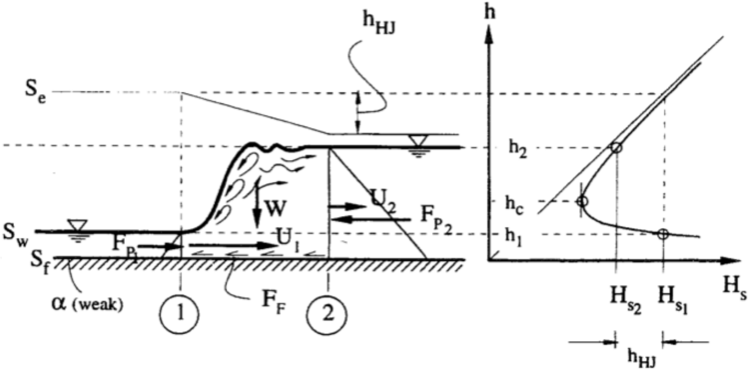
(c) Fr =2.5 to 4.5: oscillating jump;

(d) Fr =4.5 to 9.0: steady jump;

(e) Fr =9.0: strong jump.

From Moshin Siddique

Hydraulic jump: conjugate depth



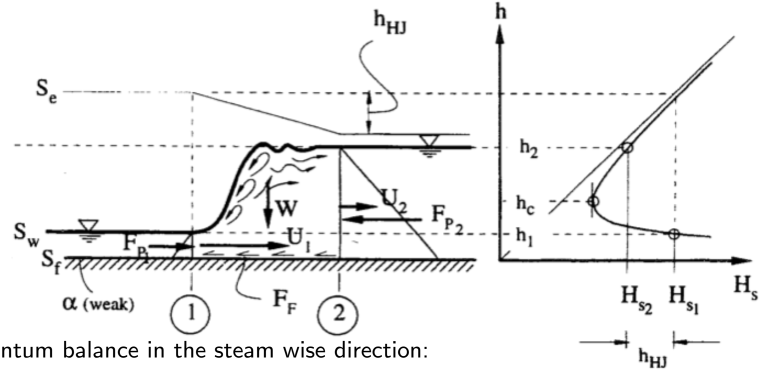
Applying the momentum balance on the control volume:

$$\int_\Omega \rho(\overrightarrow{V}.\overrightarrow{\nabla})\overrightarrow{V}\ d\Omega = -\int_\Omega \overrightarrow{\nabla} p\ d\Omega + \int_\Omega \overrightarrow{\nabla}\overline{\overline{\tau}}\ d\Omega + \int_\Omega \rho\overrightarrow{g}\ d\Omega$$

neglecting the gravity and the friction terms and applying the divergence theorem:

$$\int_\Gamma \rho\overrightarrow{V}(\overrightarrow{V}.\overrightarrow{n})\ d\Gamma = -\int_\Gamma p.\overrightarrow{n}\ d\Gamma \tag{1}$$

Hydraulic jump: conjugate depth



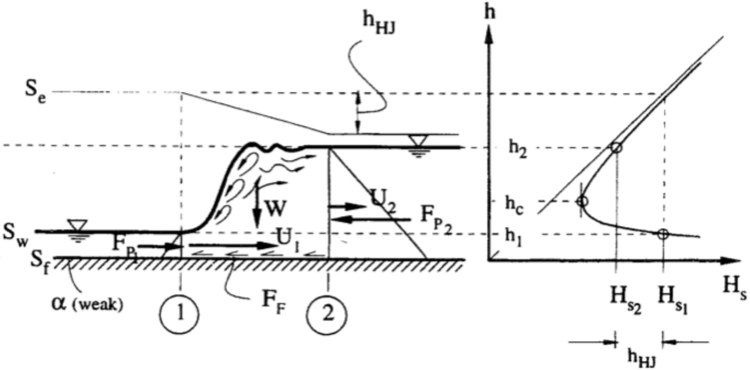
Momentum balance in the steam wise direction:

$$\int_\Gamma \rho \overrightarrow{u}(\overrightarrow{u}.\overrightarrow{n}).\overrightarrow{e_x}\; d\Gamma = -\int_\Gamma p.\overrightarrow{n}.\overrightarrow{e_x}\; d\Gamma$$

On Γ_2 and Γ_1: $\overrightarrow{n} = \pm\overrightarrow{e_x}$: $\displaystyle\int_{\Gamma_1} \rho \overrightarrow{u}(\overrightarrow{u}.\overrightarrow{n}).\overrightarrow{e_x} d\Gamma = -B\int_0^{h_1} \rho u^2 dz = -\rho B U_1^2 h_1 = -\rho\frac{Q^2}{B\ h_1}$

and $\displaystyle\int_{\Gamma_2} \rho \overrightarrow{u}(\overrightarrow{u}.\overrightarrow{n}).\overrightarrow{e_x} d\Gamma = \rho\frac{Q^2}{B\ h_2}$

Hydraulic jump: conjugate depth



On Γ_2 and Γ_1: $\overrightarrow{n} = \pm\overrightarrow{e_x}$: $\displaystyle\int_{\Gamma_1} p\overrightarrow{n}.\overrightarrow{e_x} d\Gamma = -B\int_0^{h_1} \rho\ g(h_1 - z)dz = -B\rho\ g\ \frac{h_1^2}{2}$

$$\text{and } \int_{\Gamma_2} p\overrightarrow{n}.\overrightarrow{e_x} d\Gamma = B\rho\ g\ \frac{h_2^2}{2}$$

The momentum balance is therefore: $\boxed{\rho\frac{Q^2}{B}\left(\frac{1}{h_2} - \frac{1}{h_1}\right) = B\rho\ g\ \left(\frac{h_1^2}{2} - \frac{h_2^2}{2}\right)}$

Hydraulic jump: conjugate depth

Momentum balance: $\boxed{\rho \, \frac{Q^2}{B} \left(\frac{1}{h_2} - \frac{1}{h_1} \right) = B \rho \; g \; \left(\frac{h_1^2}{2} - \frac{h_2^2}{2} \right)}$

$$2 \frac{Q^2}{g \; B^2} \frac{\cancel{h_1 - h_2}}{h_2 \; h_1} = \cancel{(h_1 - h_2)} \; (h_1 + h_2)$$

$$2 \frac{Q^2}{g \; B^2 \; h_1^3} = \frac{h_2 \; h_1}{h_1^2} \left(1 + \frac{h_2}{h_1} \right)$$

$$2 Fr_1^2 = \frac{h_2}{h_1} \left(1 + \frac{h_2}{h_1} \right)$$

This is a second order equation in $Y = \frac{h_2}{h_1}$: $Y^2 + Y - 2Fr_1^2 = 0$

of which the solutions are: $Y_{1/2} = \frac{-1 \pm \sqrt{1 + 8 \; F_1^2}}{2}$ $\qquad$ $(\; F_1 > 0 \Rightarrow 1 + 8 \; F_1^2 > 1)$

Only the plus solution is physically admissible $\boxed{\frac{h_2}{h_1} = \frac{-1 + \sqrt{1 + 8 \; F_1^2}}{2}}$

fully symmetric: $\boxed{\frac{h_1}{h_2} = \frac{-1 + \sqrt{1 + 8 \; F_2^2}}{2}}$

This equation is known as the Bélanger equation

Hydraulic jump: Head loss

Head loss definition: $\Delta H = H_2 - H_1 = \left(h_2 + \dfrac{Q^2}{2\ gB^2 h_2^2} \right) - \left(h_1 + \dfrac{Q^2}{2\ gB^2 h_1^2} \right)$

$$\Delta H = (h_2 - h_1) + \frac{Q^2}{2\ gB^2} \left(\frac{1}{h_2^2} - \frac{1}{h_1^2} \right)$$

$$\Delta H = (h_2 - h_1) + \frac{h_2\ h_1}{4} (h_1 + h_2)\ \frac{h_1^2 - h_2^2}{h_2^2\ h_1^2}$$

$$\Delta H = (h_2 - h_1) + \frac{1}{4} (h_1 + h_2)\ \frac{(h_1 - h_2)(h_1 + h_2)}{h_2\ h_1}$$

$$\Delta H = (h_1 - h_2) \left(-1 + \frac{1}{4} \frac{(h_1 + h_2)^2}{h_2\ h_1} \right)$$

$$\Delta H = (h_1 - h_2)\ \frac{-4\ h_2\ h_1 + h_1^2 + 2\ h_1\ h_2 + h_2^2}{4\ h_2\ h_1}$$

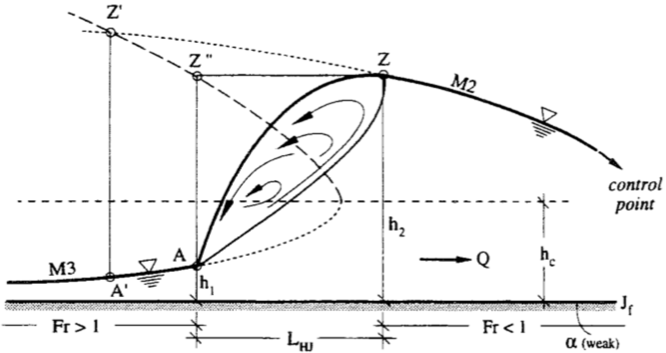
$$\Delta H = (h_1 - h_2)\ \frac{h_1^2 - 2\ h_1\ h_2 + h_2^2}{4\ h_2\ h_1}$$

$$\Delta H = (h_1 - h_2)\ \frac{(h_1 - h_2)^2}{4\ h_2\ h_1}$$

Head loss across a Hydraulic Jump: $\boxed{\Delta H = \dfrac{(h_1 - h_2)^3}{4\ h_2\ h_1}}$

valid for a rectangular cross section and an horizontal channel

Hydraulic jump: conclusion



The hydraulic jump length can be estimated as: $L_{HJ} \approx 6.1\ h_2$ for $4.5 < Fr_1 < 13$
(Henderson, 1966)

Hydraulic Jumps are usually used in hydraulic engineering as an **energy dissipator** (for example after a Dam). To prevent erosion in case of mobile beds, a still basin is build. This is the work of the hydraulic engineer to determine when and where the Hydrualic Jumps will form and to design the adequate still basin.

Hydraulic jump: Numerical Application

An hydraulic jump is used to determine the flow rate flowing in a rectangular cross-section channel of width $b = 10$ m with downstream water depth $h_1 = 0.5$ m and downstream water depth $h_2 = 1.5$ m.

1. Compute the flow rate Q
2. Compute the values of the upstream and downstream Froude numbers.
3. Compute the power dissipated by this hydraulic jump.

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