Particulate two-phase flow modelling

Application to bed-load transport in laminar shearing flows

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Acknowledgments

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Malika Ouriemi

Overview of the lecture

1. Derivation of the two-phase equations and the closure issue

   $\longrightarrow$ *From the average definition to the closed set of equations*

   The first part of this lecture is essentially based on Jackson's book (2000).

2. Application to bed-load transport in laminar shearing flows

   $\longrightarrow$ *From analytical calculation to 3D numerical model*

What are particulate two-phase flows?

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Two-phase flows are flows that involves two phases (liquid, solid, gaz).

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On the nature of the phases:
→ Liquid-Liquid, Gaz-Liquid, Gaz-Solid and **Solid-Liquid**.

On the nature of the spatial distribution of the interfaces:
→ Dispersed, Separated or Transient.

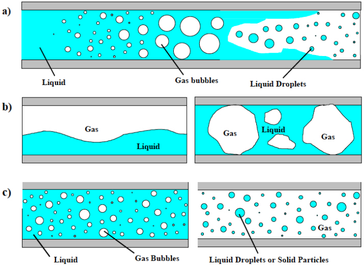

Fig.: Different regimes of two-phase flows, a) transient two-phase flow, b) separated two-phase flow, c) dispersed two-phase flow (From Sommerfeld 2000).

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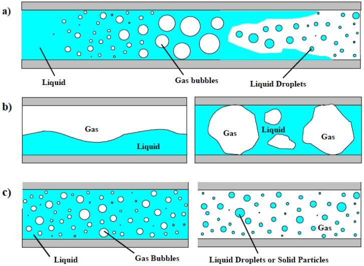
This lecture is dedicated to **dispersed Solid-Liquid** flows
→ Particulate Two-phase Flows



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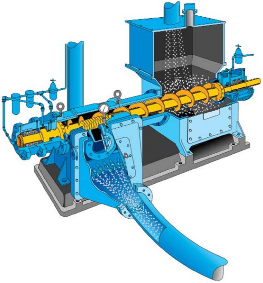
Some industrial examples

Cyclones



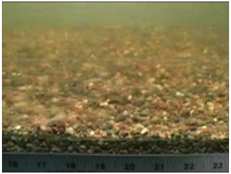
Pneumatic transport

Fluidized beds

Hoppers and bunkers

Slurry pipes

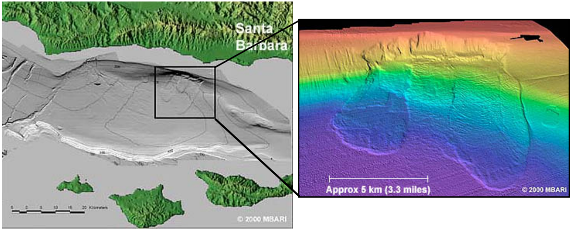
Some geophysical examples


Sediment transport


Continental Landslides (El Salvador)

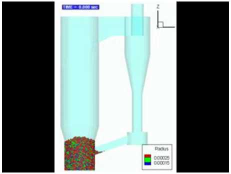

Coastal Landslides (Canada)


Snow Avalanches (Scotland)



Submarine Avalanches (Santa Barbara)

Modelling approaches?



Idea 1: Eulerian - Lagrangian approach

$\longrightarrow$ Fluid flow around each particle solved explicitely
$\Rightarrow$ Resultant force and torque exerted on each particle

$\longrightarrow$ Limited to small number of particles

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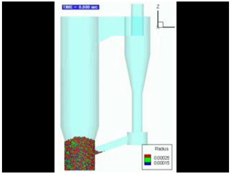

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→ Fluid flow around each particle solved explicitely
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Idea 2: Eulerian - Lagrangian approach

→ Fluid velocity spatially averaged $V_{average} >> V_{particle}$
⇒ $F_{fluid \rightarrow particle} = f(\phi, \overrightarrow{u_r})$ empirical correlations

→ Particle-particle interactions explicitely solved

"Discrete Particle Modelling" ($\geq$1990's)

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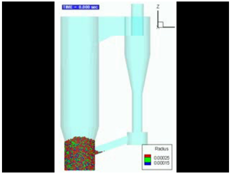

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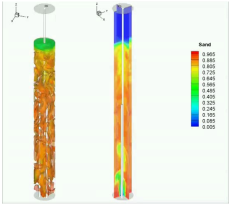

Idea 3: Eulerian - Eulerian approach

$\longrightarrow$ Fluid and particles velocities spatially averaged $V_{average} >> V_{particle}$
$\Rightarrow F_{fluid \rightarrow particle} = f(\phi, \overrightarrow{u_r})$ empirical correlations

$\longrightarrow$ No limitation on the number of particles

"Two-fluid model"

2 arguments for an Eulerian approach

- Their are approximately 1 million particles of 1 mm diameter in a cube of volume 10 cm^3 filled at 60%!

$$With \quad \begin{cases} d_p = 10^{-3}\ m \\ L = 10^{-1}\ m \\ \phi = 0.6 \end{cases} \quad we\ get \quad \begin{cases} V_t = L^3 = 10^{-3}\ m^3 \\ v_p = \dfrac{\pi}{6} d_p^3 \approx 5.10^{-10}\ m^3 \\ N_p = \dfrac{\phi V_t}{v_p} \approx 10^6\ particles \end{cases}$$

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- The solution of the Lagrangian model *would provide more detailed information than it is usually needed. Indeed, a knowledge of the average values of the velocity of the fluid, the velocities and angular velocities of the particles, and the fluid pressure, over some appropriately small region in the neighbourhood of each point [...], is usually all that is required.*

Jackson (1997)

$\Rightarrow$ **Eulerian - Eulerian approach** (Idea 3)

Objectives

Understanding the two-phase flow equations from derivation to closure

- How does one obtain the **two-phase equations**?

  $\Rightarrow$ Averaging procedure

- How does one **close the system** of partial differential equations obtained by the averaging procedure depending on the dilute or dense flow regime?

  $\Rightarrow$ Closure issue

$\longrightarrow$ Application: Bed-load transport by laminar shearing flows

Eulerian two-phase approach

$\boxed{\text{Local and instantaneous conservation equations}}$

$\Downarrow$

Averaging process

$\Downarrow$

$\boxed{\text{Averaged conservation equations for each phase}}$

$\Downarrow$

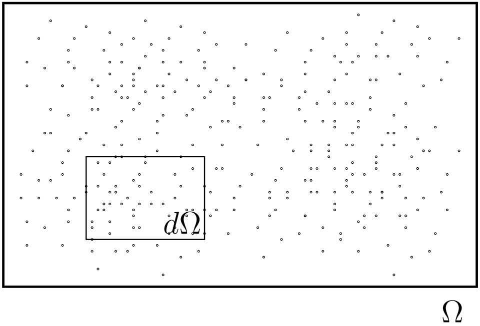
Closures

$\Downarrow$

$\boxed{\text{Closed set of equations}}$

Eulerian two-phase approach

Fluid + particles



Idea: Describe the motion of the fluid and the particles by a continuous approach at the scale of $d\Omega$ (*i.e.*:$D_{particle} << L_{d\Omega}$)

Various averaging approach

- **Ensemble average**

  → Averaged at each point of space over an ensemble of ''macroscopically" equivalent systems

  Drew and Lahey (1993), Zhang and Prosperetti (1997), ...

- **Local spatial average**

  → Averaged over small region compared to macroscopic length scale of interest

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- **Local spatial average**

  $\longrightarrow$ Averaged over small region compared to macroscopic length scale of interest

  Jackson (1997 and 2000), ...

$\longrightarrow$ The two averaging procedures are equivalent and led to the same equations

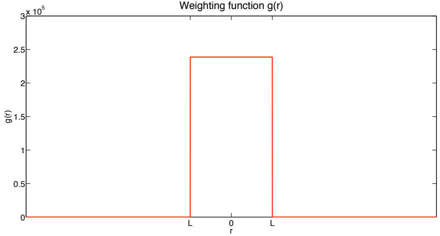
$\Rightarrow$ **We describe here the local spatial average**

Definition of the local space average (Jackson, 1997 and 2000)

Weighting function $g(r)$

$$g(r) = \begin{cases} \dfrac{3}{4\ \pi\ L^3} & for \quad r \leq L \\ 0 & for \quad r > L \end{cases}$$

Verifying $\quad 4\pi \int_0^\infty g(r) r^2 dr = 1$



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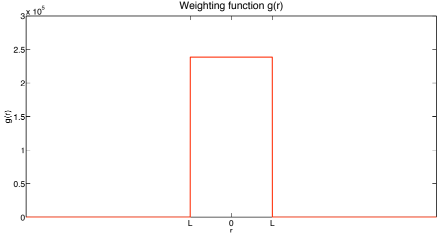
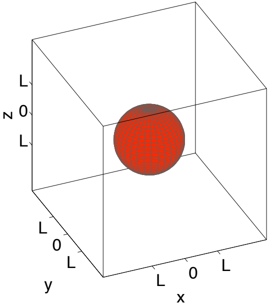


Overall average value of f at position $\overrightarrow{x}$ and time t: $\langle f \rangle$

$$\langle f \rangle (\overrightarrow{x}, t) = \int_V f(\overrightarrow{y}, t)\ g(|\overrightarrow{x} - \overrightarrow{y}|) dV_y$$

L must be chosen such that: $L_{macro} >> L >> D_p$

$\Rightarrow$ Separation of scales: $L_{macro} >> D_p$



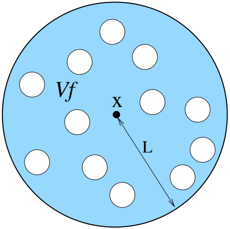
Definition of the fluid phase average

Void fraction ϵ

$\rightarrow$ fraction of space occupied by fluid in the neighbourhood of $\overrightarrow{x}$

$$\epsilon(\overrightarrow{x},t) = \int_{V_f(t)} g(|\overrightarrow{x} - \overrightarrow{y}|) dV_y = \frac{V_f}{V}$$

where $V_f(t)$ indicate the part of the system occupied by fluid at time t.



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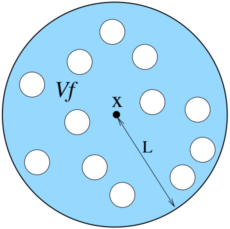


Fluid phase average of f at position $\overrightarrow{x}$ and time t: $\langle f \rangle^f$

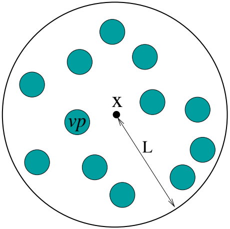
$$\epsilon(\overrightarrow{x},t)\langle f \rangle^f(\overrightarrow{x},t) = \int_{V_f(t)} f(\overrightarrow{y},t) g(|\overrightarrow{x} - \overrightarrow{y}|)dV_y$$

Definition of the solid phase average

Solids volume fraction ϕ

$\rightarrow$ fraction of space occupied by particles in the neighbourhood of $\overrightarrow{x}$

$$\phi(\overrightarrow{x},t) = \sum_p \int_{v_p} g(|\overrightarrow{x} - \overrightarrow{y}|) dv_y = \frac{\sum_p v_p}{V}$$



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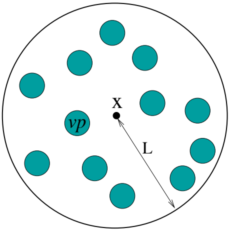


Solid phase average of f at position $\overrightarrow{x}$ and time t: $\langle f \rangle^s$

$$\phi(\overrightarrow{x},t) \langle f \rangle^s (\overrightarrow{x},t) = \sum_p \int_{v_p} f(\overrightarrow{y},t) g(|\overrightarrow{x} - \overrightarrow{y}|)dv_y$$

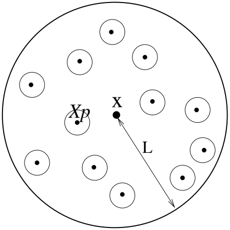
Definition of the particle phase average

Number density n

→Number of particles / unit volume in the neighbourhood of $\overrightarrow{x}$

$$n(\overrightarrow{x}) = \sum_p g(|\overrightarrow{x} - \overrightarrow{x_p}|)$$

where $\overrightarrow{x_p}$ is the position of the centre of particle p.



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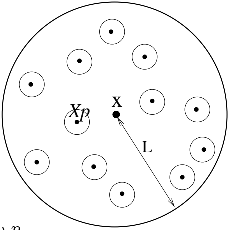

Particle phase average of $\mathbf{f}$ at position $\overrightarrow{x}$ and time t: $\langle f \rangle^p$

$$n(\overrightarrow{x},t) \langle f \rangle^p (\overrightarrow{x},t) = \sum_p f^p g(|\overrightarrow{x} - \overrightarrow{x_p}|)$$

Link with the solid phase average $\boxed{n(\overrightarrow{x},t)\ v_p = \phi(\overrightarrow{x},t)}$.

Theorems

Using **Leibniz rule** we can demonstrate that (differentiation under the integral sign)

Theorem 1 ($Th1$):

$$\epsilon\left\langle\frac{\partial f}{\partial x_k}\right\rangle^f=\frac{\partial\epsilon\left\langle f\right\rangle^f}{\partial x_k}-\sum_p\int_{s_p}f(\overrightarrow{y})n_k(\overrightarrow{y})g(|\overrightarrow{x}-\overrightarrow{y}|)ds_y$$

Theorem 2 ($Th2$):

$$\epsilon\left\langle\frac{\partial f}{\partial t}\right\rangle^f=\frac{\partial\epsilon\left\langle f\right\rangle^f}{\partial t}+\sum_p\int_{s_p}f(\overrightarrow{y})n_k(\overrightarrow{y})u_k(\overrightarrow{y})g(|\overrightarrow{x}-\overrightarrow{y}|)ds_y$$

$\rightarrow$ The same relationship exists for the solid phase average

Theorem 3 ($Th3$):

$$n\left\langle\frac{\partial f}{\partial t}\right\rangle^p=\frac{\partial n\left\langle f\right\rangle^p}{\partial t}+\frac{\partial}{\partial x_k}\sum_p f^p u_k^p g(|\overrightarrow{x}-\overrightarrow{x_p}|)$$

Local and instantaneous conservation equations

- General continuity equation

$$\overrightarrow{\nabla}.\overrightarrow{u} = 0$$

- General momentum equation

$$\rho \left[\frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{\nabla}.\left(\overrightarrow{u} \otimes \overrightarrow{u}\right) \right] = \overrightarrow{\nabla}.\left(\overline{\overline{\sigma}}\right) + \rho \overrightarrow{g}$$

Applying the spatial averaging for the fluid and the solid phases separetly to these conservation equations we obtain the two-phase equations

Average equation of fluid mass conservation

1. Continuity equation for the mixture: $\dfrac{\partial u_k}{\partial x_k} = 0$

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3. Using previous theorems

$$\begin{cases} Th1: \; f = u_k \;\; \rightarrow \;\; \epsilon \left\langle \dfrac{\partial u_k}{\partial x_k} \right\rangle^f = \dfrac{\partial \epsilon \left\langle u_k \right\rangle^f}{\partial x_k} - \displaystyle\sum_p \int_{s_p} u_k(\overrightarrow{y}) n_k(\overrightarrow{y}) g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y \\ Th2: \; f = 1 \;\; \rightarrow \;\; 0 = \dfrac{\partial \epsilon \left\langle u_k \right\rangle^f}{\partial t} + \displaystyle\sum_p \int_{s_p} n_k(\overrightarrow{y}) u_k(\overrightarrow{y}) g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y \end{cases}$$

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4. Substituting the term $\sum_p \int_{s_p} \ldots ds_y$ from the second equation in the first one we obtain the mass conservation equation for the fluid phase

$$\boxed{\frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon \left\langle u_k \right\rangle^f}{\partial x_k} = 0}$$

Average equation of fluid momentum

1. Point momentum equation for the fluid: $\rho_f \left[\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_k}{\partial x_k} \right] = \frac{\partial \sigma_{ik}}{\partial x_k} + \rho_f g_i$

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4. Fluid phase momentum equation

$$\boxed{\rho_f \left[\frac{\partial \epsilon \left\langle u_i \right\rangle^f}{\partial t} + \frac{\partial \epsilon \left\langle u_i u_k \right\rangle^f}{\partial x_k}\right] = \frac{\partial \epsilon \left\langle \sigma_{ik} \right\rangle^f}{\partial x_k} - \sum_p \int_{s_p} \sigma_{ik}(\vec{y}) n_k(\vec{y}) g(|\vec{x} - \vec{y}|) ds_y + \epsilon \rho_f g_i}$$

Averaged equations for the fluid and solid phase

Fluid phase equations

Mass
$$\frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon \left\langle u_k \right\rangle^f}{\partial x_k} = 0$$

Momentum

$$\rho_f \left[\frac{\partial \epsilon \left\langle u_i \right\rangle^f}{\partial t} + \frac{\partial \epsilon \left\langle u_i u_k \right\rangle^f}{\partial x_k} \right] = \frac{\partial \epsilon \left\langle \sigma_{ik} \right\rangle^f}{\partial x_k} - \sum_p \int_{s_p} \sigma_{ik} n_k g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y + \epsilon \rho_f g_i$$

Averaged equations for the fluid and solid phase

Fluid phase equations

Mass

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon \left\langle u_k \right\rangle^f}{\partial x_k} = 0$$

Momentum

$$\rho_f \left[\frac{\partial \epsilon \left\langle u_i \right\rangle^f}{\partial t} + \frac{\partial \epsilon \left\langle u_i u_k \right\rangle^f}{\partial x_k} \right] = \frac{\partial \epsilon \left\langle \sigma_{ik} \right\rangle^f}{\partial x_k} - \sum_p \int_{s_p} \sigma_{ik} n_k g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y + \epsilon \rho_f g_i$$

Solid phase equations

Mass

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi \left\langle u_k \right\rangle^s}{\partial x_k} = 0$$

Momentum

$$\rho_s \left[\frac{\partial \phi \left\langle u_i \right\rangle^s}{\partial t} + \frac{\partial \phi \left\langle u_i u_k \right\rangle^s}{\partial x_k} \right] = \frac{\partial \phi \left\langle \sigma_{ik} \right\rangle^s}{\partial x_k} + \sum_p \int_{s_p} \sigma_{ik} n_k g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y + \phi \rho_s g_i$$

Average equation of particle momentum

1. Momentum equation for particle p:

$$\rho_f v_p \frac{\partial u_i^p}{\partial t} = \underbrace{\int_{S_p} \sigma_{ik}(\overrightarrow{y}) n_k(\overrightarrow{y}) dS_y}_{traction\ exerted\ by\ the\ fluid} + \underbrace{\sum_{q \neq p} f^{pq}}_{Contact\ forces} + \rho_s v_p g_i$$

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2. Applying the particle phase average gives

$$\rho_s v_p \left\langle \frac{\partial n u_i}{\partial t} \right\rangle^p = \sum_p g(|\overrightarrow{x} - \overrightarrow{x_p}|) \left[\int_{S_p} \sigma_{ik} n_k dS_y + \sum_{q \neq p} f^{pq} \right] + \rho_s v_p n g_i$$

3. Using Theorem 3 with $f = u_i$ and the fact that $v_p n = \phi$

$$n \left\langle \frac{\partial u_i}{\partial t} \right\rangle^p = \frac{\partial n \langle u_i \rangle}{\partial t} + \frac{\partial}{\partial x_k} \sum_p u_i u_k g(|\overrightarrow{x} - \overrightarrow{x_p}|)$$

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4. Particle phase momentum equation

$$\boxed{\rho_s \left[\frac{\partial \phi \langle u_i \rangle}{\partial t} + \frac{\partial \phi \langle u_i u_k \rangle^p}{\partial x_k} \right] = \sum_p g(|\overrightarrow{x} - \overrightarrow{x_p}|) \left[\int_{S_p} \sigma_{ik} n_k dS_y + \sum_{q\neq p} f^{pq} \right] + \rho_s \phi g_i}$$

Fluid-particle interactions

Newton's third law (action-reaction law)

$$\sum_p g(|\overrightarrow{x} - \overrightarrow{x_p}|) \int_{S_p} \sigma_{ik} n_k dS_y = \sum_p \int_{s_p} \sigma_{ik} n_k g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y$$

and using a Taylor expansion of the weighting function

$$g(|\overrightarrow{x} - \overrightarrow{y}|) = g(|\overrightarrow{x} - \overrightarrow{x_p}|) - a \frac{\partial g(|\overrightarrow{x} - \overrightarrow{x_p}|)}{\partial x_j} n_j - \frac{a^2}{2} \frac{\partial^2 g(|\overrightarrow{x} - \overrightarrow{x_p}|)}{\partial x_j \partial x_k} n_j n_k + \ldots$$

we can demonstrate that

$$\boxed{\sum_p \int_{s_p} \sigma_{ik}(\overrightarrow{y}) n_k(\overrightarrow{y}) g(|\overrightarrow{x} - \overrightarrow{y}|) ds_y = n \left\langle f_i^f \right\rangle^p - \frac{\partial n \left\langle s_{ij}^f \right\rangle^p}{\partial x_j} + \ldots}$$

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Interpretation

- $n \left\langle f_i^f \right\rangle^p$: Average of the integral of the fluid stress on the surface of each particle
  $\Rightarrow$ Forces exerted by the fluid on the particles
- $\dfrac{\partial n \left\langle s_{ij}^f \right\rangle^p}{\partial x_j}$: Effect of the presence of the particles on the stresses
  $\longrightarrow$ Effective stress tensor $S_{ik}^f = \epsilon \left\langle \sigma_{ik} \right\rangle^f + n \left\langle s_{ij}^f \right\rangle^p + \ldots$

Two-phase equations

Simplified notations :

$\langle \vec{u} \rangle^f \longrightarrow \vec{u}^f$

$\langle \vec{u} \rangle^s \longrightarrow \vec{u}^s$

$n \left\langle \vec{f}^f \right\rangle^p \longrightarrow n \vec{f}$

Mass conservation equations

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla}.\left(\epsilon \overrightarrow{u^f}\right) = 0 \quad ; \quad \frac{\partial \phi}{\partial t} + \vec{\nabla}.\left(\phi \overrightarrow{u^p}\right) = 0$$

Momentum conservation equations

$$\rho_f \left[\frac{\partial \epsilon \overrightarrow{u^f}}{\partial t} + \vec{\nabla}.\left(\epsilon \overrightarrow{u^f} \otimes \overrightarrow{u^f}\right)\right] = \vec{\nabla}.\left(\overline{\overline{S^f}}\right) - n \vec{f} + \epsilon \rho_f \vec{g}$$

$$\underbrace{\rho_s \left[\frac{\partial \phi \overrightarrow{u^p}}{\partial t} + \vec{\nabla}.\left(\phi \overrightarrow{u^p} \otimes \overrightarrow{u^p}\right)\right]}_{Inertia} = \underbrace{\vec{\nabla}.\left(\overline{\overline{S^p}}\right)}_{Stresses} + \underbrace{n \vec{f}}_{Interaction} + \underbrace{\phi \rho_s \vec{g}}_{Gravity}$$

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We need to relate the fluid and solid phase stress tensors $\overline{\overline{S^f}}$, $\overline{\overline{S^p}}$ and the interaction term $n \overrightarrow{f}$ to the average variables $\epsilon, \phi, \overrightarrow{u^f}, \overrightarrow{u^s}$

Remarks on mixture average equations

We can **substitute** one of the **fluid or solid** conservation **equation** by a **mixture** conservation **equation**.

Definitions

- Volume average velocity: $\langle \overrightarrow{u} \rangle = \epsilon \langle \overrightarrow{u} \rangle^f + \phi \langle \overrightarrow{u} \rangle^s$
- Global volume conservation $\Rightarrow \epsilon + \phi = 1$ *i.e.:* $V = V_f + V_p$

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Summing the mass conservation equations for the two phases gives

$$\frac{\partial \overbrace{\epsilon + \phi}^{=1}}{\partial t} + \overrightarrow{\nabla}.\overbrace{(\epsilon \overrightarrow{u^f} + \phi \overrightarrow{u^p})}^{=\langle u_k \rangle} = 0 \Rightarrow \boxed{\frac{\partial \langle u_k \rangle}{\partial x_k} = 0}$$

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$\Rightarrow$ The **mixture is incompressible** in terms of the volume average velocity

Definitions

- Mixture density: $\rho^m = \epsilon\rho_f + \phi\rho_s$
- Mass average velocity: $\rho^m \left\langle \overrightarrow{u} \right\rangle^m = \epsilon\rho_f \left\langle \overrightarrow{u} \right\rangle^f + \phi\rho_s \left\langle \overrightarrow{u} \right\rangle^s$

Definitions

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Averaging the point conservation equation over the whole space led to the following equations

Mass conservation of the mixture

$$\frac{\partial \rho^m}{\partial t} + \frac{\partial \rho^m \langle u_k \rangle^m}{\partial x_k} = 0$$

Momentum equation for the mixture

$$\rho^m \left[\frac{\partial \langle u_i \rangle^m}{\partial t} + \frac{\partial \langle u_i \rangle^m \langle u_k \rangle^m}{\partial x_k} \right] = \frac{\partial S_{ik}^m}{\partial x_k} + \rho^m g_i$$

Stress tensor: $S_{ik}^m = S_{ik}^f + S_{ik}^s - \dfrac{\epsilon\phi\rho_s\rho_f}{\rho^m} \left(\langle u_i \rangle^f - \langle u_i \rangle^s \right) \left(\langle u_k \rangle^f - \langle u_k \rangle^s \right)$

Position of the problem

From the derivation of the conservation equations we get:

- 29 unknowns
  - $\longrightarrow$ 2 volume fractions ϵ,ϕ
  - $\longrightarrow$ 2x3 velocity components u_i^f,u_i^s
  - $\longrightarrow$ 2x9 stress tensor components S_{ik}^f,S_{ik}^s
  - $\longrightarrow$ 3 interaction term components $n\ f_i$
- 8 equations
  - $\longrightarrow$ 2 mass conservation equations
  - $\longrightarrow$ 2x3 momentum equations

$\Rightarrow$ We need 21 additional equations !!!

Hypothesis

1. We neglect the Reynolds like contribution $Re \leq 1$

$$\Rightarrow \langle u_i u_k \rangle^f = \langle u_i \rangle^f \langle u_k \rangle^f$$

2. The fluid is Newtonian

$$\Rightarrow \sigma_{ik} = -p + \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

3. The particles are spherical, rigid and monodisperse

The closure consists in expressing $\langle \sigma_{ik} \rangle^f$, S_{ik}^f, S_{ik}^s and $n \left\langle f_i^f \right\rangle^p$ in terms of local average variables and their derivatives.

Closure for the fluid stress tensors

1. Newtonian fluid: $\sigma_{ik} = -p\delta_{ik} + \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$

2. Averaging over the fluid phase: $\epsilon \left\langle \sigma_{ik} \right\rangle^f = -\epsilon \left\langle p \right\rangle^f + \eta\epsilon \left\langle \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right\rangle^f$

   with $\epsilon \left\langle \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right\rangle^f = \frac{\partial \left\langle u_i \right\rangle}{\partial x_k} + \frac{\partial \left\langle u_k \right\rangle}{\partial x_i}$

   $\int_{V_f} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) g(\overrightarrow{r}) dV_y = \int_V \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) g(|\overrightarrow{x} - \overrightarrow{y}|) dV_y$

3. Fluid average stress tensor: $\epsilon \left\langle \sigma_{ik} \right\rangle^f = -\epsilon \left\langle p \right\rangle^f + \eta \left(\frac{\partial \left\langle u_i \right\rangle}{\partial x_k} + \frac{\partial \left\langle u_k \right\rangle}{\partial x_i} \right)$

   expressed in terms of the volume average velocity $\left\langle \overrightarrow{u} \right\rangle$

Closure for the effective fluid stress tensors

We simply introduce an effective viscosity η_{eff} in the previous definition that takes into account the effect of the presence of particles on the shear resistance of the fluid

$$\boxed{S^f_{ik} = -\epsilon \langle p \rangle^f \delta_{ik} + \eta_{eff} \left(\frac{\partial \langle u_i \rangle}{\partial x_k} + \frac{\partial \langle u_k \rangle}{\partial x_i} \right)}$$

Dilute flows: $\boxed{\eta_{eff} = \eta \left(1 + \frac{5}{2} \phi \right)}$ **Einstein (1906)**

Valid for $\phi < 0.3\%$

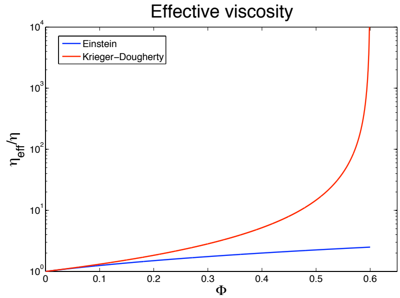
$\rightarrow$ We have 9 new equations, we need 12 additional equations

Closure for the effective fluid stress tensors

Dense flows: $$\boxed{\eta_{eff} = \eta \left(1 - \frac{\phi}{\phi_{max}}\right)^{-\frac{5}{2}\phi_{max}}}$$ **Krieger-Dougherty (19??)**

Imagine the case of a fixed assembly of particles close to maximum packing $\phi \approx \phi_{max}$ then $\eta_{eff} \to \infty$.

$\Rightarrow$ The fluid cannot be sheared !!!



Krieger-Dougherty's viscosity has been developed for the mixture as a whole without differentiation between fluid and particles behaviour

Closure for the effective solid stress tensors : Particle-particle interactions

- **In very dilute suspension** $\overline{\overline{S^s}} \approx 0$

  $\longrightarrow$ No contact between particles.

- **When** ϕ **increases** $\overline{\overline{S^s}} \neq 0$

  $\longrightarrow$ Collisions between particles occurs $\Rightarrow$ Kinetic and collisional stresses

- **When** $\phi \rightarrow \phi_{max}$

  $\longrightarrow$ Enduring contact between particles exists $\Rightarrow$ Frictional stresses

  See the lecture from O. Pouliquen this morning

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General stress-shear rate relation:

$$\boxed{S^s_{ik} = - \langle p \rangle^s \delta_{ik} + \eta^s_{eff} \left(\frac{\partial \langle u_i \rangle^s}{\partial x_k} + \frac{\partial \langle u_k \rangle^s}{\partial x_i} \right)}$$

where $\langle p \rangle^s$ and η^s_{eff} depends on the physics at work

$\longrightarrow$ We have 9 new equations, we need 3 more equations

Collisional regime

Granular material $\approx$ Molecular gas

$\neq$ Inelastic nature of collisions

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Kinetic theory of Granular flows

$\rightarrow$ *Haff (1983), Jenkins and Savage (1983), Lun et al. (1984), ...*

$$\Rightarrow \boxed{\langle p\rangle^s = \langle p\rangle^s(\phi, T^s) \text{ and } \eta^s_{eff} = \eta^s_{eff}(\phi, T^s)}$$

where $T^s = \langle u_i^s u_k^s\rangle^p$ is the "particle temperature"or the "pseudothermal" energy

$\Rightarrow$ An additional equation for T^s needs to be solved.

See Koch and coworkers, Buyevich and coworkers or Simonin and coworkers for further details

Frictional regime

Frictional rheology: $\mu(I)$

$$\boxed{S^s_{ik} = -\left\langle p \right\rangle^s \delta_{ik} + \eta^s_{eff} \left(\frac{\partial \left\langle u_i \right\rangle^s}{\partial x_k} + \frac{\partial \left\langle u_k \right\rangle^s}{\partial x_i} \right)}$$

$\rightarrow$ *GDR Midi (2004), Jop et al. (2006), Forterre and Pouliquen (2008)*

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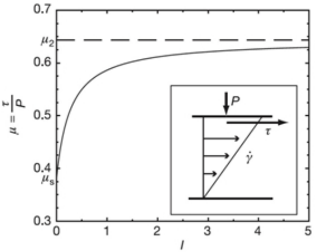

From Jop et al. (2006)

Where $\eta^s_{eff} = \dfrac{\mu(I)\left\langle p\right\rangle^s}{\| \overline{\overline{\dot{\gamma}^s}} \|}$

and $\mu(I) = \mu_s + \dfrac{\mu_2 - \mu_s}{\dfrac{I_0}{I} + 1}$

with I the inertial parameter.

Simplified frictional rheology (Coulomb):
$\rightarrow \mu = \mu_s = constant$

Fluid-particle interactions $n\overrightarrow{f}$

Following Jackson (2000) we can write $n\overrightarrow{f}$ as

$$n\overrightarrow{f} = \underbrace{\phi\overrightarrow{\nabla}.\overline{\overline{S^f}}}_{Buoyancy} + n\overrightarrow{f_1} \quad \text{or} \quad n\overrightarrow{f} = -\rho_f\phi\underbrace{\left(\overrightarrow{g} - \frac{D_f\overrightarrow{u^f}}{Dt}\right)}_{Specific\ gravity} + n\overrightarrow{f_2}$$

with $\overrightarrow{f_2} = \dfrac{\overrightarrow{f_1}}{\epsilon}$

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$$\text{with } \overrightarrow{f_2} = \frac{\overrightarrow{f_1}}{\epsilon}$$

<u>Remark</u>

"When a fluid flows through an assembly of particle there is a contribution to its local average pressure gradient from the relative motion. But this contribution, in turn, exerts a force on the immersed particles that may be attributed either to the buoyancy term [$\phi\overrightarrow{\nabla}.\overline{\overline{S^f}}$] or to the other term in the decomposition [$n\overrightarrow{f_2}$]. [...] this is the reason why $\overrightarrow{f_2}$ is greater than $\overrightarrow{f_1}$ by a factor $1/\epsilon$."

Jackson (2000)

Two-phase equations

Using the first decomposition for the fluid-particle interaction led to

Mass conservation equations

$$\frac{\partial \epsilon}{\partial t} + \overrightarrow{\nabla}.\left(\epsilon \overrightarrow{u^f}\right) = 0 \qquad ; \qquad \frac{\partial \phi}{\partial t} + \overrightarrow{\nabla}.\left(\phi \overrightarrow{u^p}\right) = 0$$

Momentum conservation equations

$$\rho_f \left[\frac{\partial \epsilon \overrightarrow{u^f}}{\partial t} + \overrightarrow{\nabla}.\left(\epsilon \overrightarrow{u^f} \otimes \overrightarrow{u^f}\right)\right] = \overbrace{(1-\phi)}^{=\epsilon} \overrightarrow{\nabla}.\left(\overline{\overline{S^f}}\right) - n\overrightarrow{f^1} + \epsilon \rho_f \overrightarrow{g}$$

$$\rho_s \left[\frac{\partial \phi \overrightarrow{u^p}}{\partial t} + \overrightarrow{\nabla}.\left(\phi \overrightarrow{u^p} \otimes \overrightarrow{u^p}\right)\right] = \overrightarrow{\nabla}.\left(\overline{\overline{S^p}}\right) + \phi \overrightarrow{\nabla}.\left(\overline{\overline{S^f}}\right) + n\overrightarrow{f^1} + \phi \rho_s \overrightarrow{g}$$

Fluid-particle interaction force $n\overrightarrow{f_1}$

$$\text{From experimental evidence}: n\overrightarrow{f_1} = \overbrace{\overrightarrow{F_D}}^{Drag} + \overbrace{\overrightarrow{F_L}}^{Lift} + \overbrace{\overrightarrow{F_{VM}}}^{Virtual\ Mass} + \ldots$$

- Drag force: $\boxed{\overrightarrow{F_D} = F(\phi, |\overrightarrow{u^f} - \overrightarrow{u^s}|)\ \left(\overrightarrow{u^f} - \overrightarrow{u^s}\right)}$

  $\longrightarrow$ Colinear to the relative motion

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- Lift force: $\boxed{\overrightarrow{F_L} = C_L(\phi)\ \rho_f\ \phi\ \left(\overrightarrow{\nabla} \wedge \overrightarrow{u^f}\right) \wedge \left(\overrightarrow{u^f} - \overrightarrow{u^s}\right)}$

  $\longrightarrow$ Perpendicular to the plan formed by the relative motion and the vorticity

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  $\longrightarrow$ Perpendicular to the plan formed by the relative motion and the vorticity

- Virtual Mass force: $\boxed{\overrightarrow{F_{VM}} = C_{VM}(\phi)\ \rho_f\ \phi\ \left(\frac{D_f\overrightarrow{u^f}}{Dt} - \frac{D_s\overrightarrow{u^s}}{Dt}\right)}$

  $\longrightarrow$ Colinear to the relative acceleration

$\longrightarrow$ We have 3 new equations, the two-phase equations are closed !

Virtual Mass force

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An accelerating or decelerating particle must move some volume of surrounding fluid with it as it moves. The virtual mass force opposes the motion of particles.

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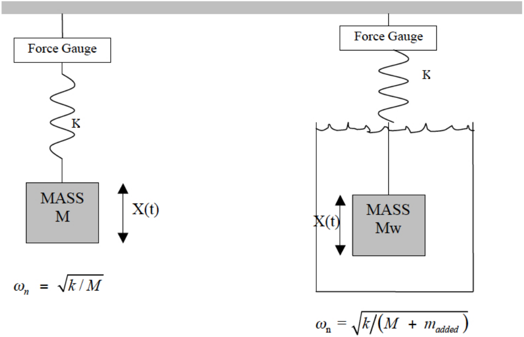
$$\overrightarrow{F_{VM}} = C_{VM}(\phi)\ \rho_f\ \phi\ \left(\frac{D_f \overrightarrow{u^f}}{Dt} - \frac{D_s \overrightarrow{u^s}}{Dt} \right)$$

An accelerating or decelerating particle must move some volume of surrounding fluid with it as it moves. The virtual mass force opposes the motion of particles.

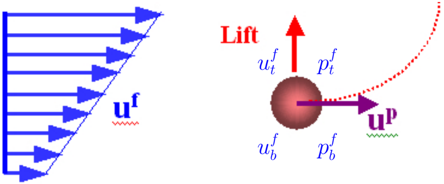


ω_n: Natural frequency

k: Spring stifness

Lift force

$$\boxed{\overrightarrow{F_L} = C_L(\phi)\ \rho_f\ \phi\ \left(\overrightarrow{\nabla} \wedge \overrightarrow{u^f}\right) \wedge \left(\overrightarrow{u^f} - \overrightarrow{u^s}\right)}$$



$$u_t^f > u_b^f \Rightarrow p_t^f < p_b^f$$

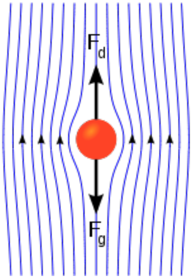
$\exists$ a force from low to high velocity

Lift force on a single particle (Saffman ,1965 and 1968):

$$F_L = C_L^{Saf.}\ \frac{\partial u^f}{\partial z}\left(u^f - u^p\right)$$

Drag force - Dilute flows

$$\boxed{\overrightarrow{F_D} = F(\phi, |\overrightarrow{u^f} - \overrightarrow{u^s}|) \left(\overrightarrow{u^f} - \overrightarrow{u^s}\right)}$$



Stokes drag on a single particle: $f_D = 3\ \pi\ D_p\ \eta^f \ \left(\overrightarrow{u^f} - \overrightarrow{u^p}\right)$

Recalling $F_D = n\ f_D$ with $n\ v_p = n \dfrac{\pi\ D_p^3}{6} = \phi$

we obtain: $F_D = 18\ \dfrac{\phi\ \eta^f}{D_p^2} \ \left(\overrightarrow{u^f} - \overrightarrow{u^p}\right)$

Dimensional analysis: $F_D = \dfrac{1}{2}\ \rho_f\ C_D\ \overbrace{\dfrac{\pi D_p^2}{4}}^{A_p}\ \left|\overrightarrow{u^f} - \overrightarrow{u^p}\right| \left(\overrightarrow{u^f} - \overrightarrow{u^p}\right)$

Identifying the two expressions we get: $C_D = \dfrac{24\eta^f}{\rho_f D_p |\overrightarrow{u^f} - \overrightarrow{u^p}|} = \dfrac{24}{Re_p}$ for $Re << 1$

Drag force - Dilute flows

- **Influence of inertial effect (transition regime)**

First order correction: $C_D = \dfrac{24}{Re_p}\left(1 + \dfrac{3}{16} Re_p\right)$ for $Re_p < 5$

Oseen (1910)

Empirical correlation: $C_D = \dfrac{24}{Re_p}\left(1 + 0.15\ Re_p^{0.687}\right)$ for $0.5 < Re_p < 1000$

Schiller and Naumann (1933)

Drag force - Dilute flows

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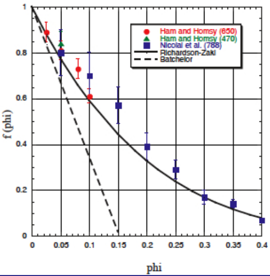
Schiller and Naumann (1933)

- **Influence of the particles concentration**

$$v_s = v_t \underbrace{(1-\phi)^n}_{=f(\phi)}$$ *Richardson-Zaki (1954)*

$$\Rightarrow F_D = n\ f_D\ (1-\phi)^{2-n}$$

$n = 4.65$ for low Reynolds Number Re_p



From Chehata at al. (2006)

Drag force - Dense flows

$$\boxed{\overrightarrow{F_D} = F(\phi, |\overrightarrow{u^f} - \overrightarrow{u^s}|) \left(\overrightarrow{u^f} - \overrightarrow{u^s}\right)}$$

Darcy drag

$$\overrightarrow{F_D} = \frac{\eta\epsilon^2}{K}\left(\overrightarrow{u^f} - \overrightarrow{u^p}\right)$$

- Fluid viscosity: η
- Permeability: $K = \frac{\epsilon^3 d^2}{k(1-\epsilon)^2}$

with $k \approx 180$: Kozeny-Karman relation for the permeability (Goharzadeh et al., 2005)

$$\Rightarrow \boxed{F(\phi, |\overrightarrow{u^f} - \overrightarrow{u^s}|) = \frac{\eta\epsilon^2}{K}}$$

Case of a suspension of particles in sedimentation

Hypothesis:

- Inertia of the fluid is negligible
- The fluid-particle interaction is dominated by the drag force

Mass conservation equations

$$\frac{\partial \epsilon}{\partial t} + \overrightarrow{\nabla}.\left(\epsilon \overrightarrow{u^f}\right) = 0 \qquad ; \qquad \frac{\partial \phi}{\partial t} + \overrightarrow{\nabla}.\left(\phi \overrightarrow{u^p}\right) = 0$$

Momentum conservation equations

$$\rho_f \frac{\partial \epsilon \overrightarrow{u^f}}{\partial t} = -\epsilon \overrightarrow{\nabla} p^f + \epsilon \overrightarrow{\nabla}.\left(\overline{\overline{\tau^f}}\right) - F(\overrightarrow{u^f} - \overrightarrow{u^p}) + \epsilon \rho_f \overrightarrow{g}$$

$$\rho_s \frac{\partial \phi \overrightarrow{u^p}}{\partial t} = -\overrightarrow{\nabla} p^p + \overrightarrow{\nabla}.\left(\overline{\overline{\tau^p}}\right) - \phi \overrightarrow{\nabla} p^f + \phi \overrightarrow{\nabla}.\left(\overline{\overline{\tau^f}}\right) + F(\overrightarrow{u^f} - \overrightarrow{u^p}) + \phi \rho_s \overrightarrow{g}$$

Case of a suspension of particles in sedimentation

Rewriting the equation for a 1D vertical problem, we get:

Mass conservation equations

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon w^f}{\partial z} = 0 \qquad ; \qquad \frac{\partial \phi}{\partial t} + \frac{\partial \phi w^p}{\partial z} = 0 \qquad ; \qquad \frac{\partial w^m}{\partial z} = 0$$

Momentum conservation equations

$$\rho_f \frac{\partial \epsilon w^f}{\partial t} = -\epsilon \frac{\partial p^f}{\partial z} + \epsilon \frac{\partial}{\partial z}(\eta_{eff} \frac{\partial w^m}{\partial z}) - F(w^f - w^p) - \epsilon \rho_f g$$

$$\rho_s \frac{\partial \phi w^p}{\partial t} = -\frac{\partial p^p}{\partial z} + \frac{\partial}{\partial z}(\eta^p \frac{\partial w^p}{\partial z}) - \phi \frac{\partial p^f}{\partial z} + \phi \frac{\partial}{\partial z}(\eta_{eff} \frac{\partial w^m}{\partial z}) + F(w^f - w^p) - \phi \rho_s g$$

Mixture momentum equation

$$\rho^m \frac{\partial w^m}{\partial t} = -\frac{\partial p^p}{\partial z} - \frac{\partial p^f}{\partial z} + \frac{\partial}{\partial z}(\eta^p \frac{\partial w^p}{\partial z}) + \frac{\partial}{\partial z}(\eta_{eff} \frac{\partial w^m}{\partial z}) - \rho^m g$$

We can choose 4 of the six previous equations

Mass conservation equations

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi w^p}{\partial z} = 0 \qquad ; \qquad \frac{\partial w^m}{\partial z} = 0$$

Particle momentum equation

$$\rho_s \frac{\partial \phi w^p}{\partial t} = -\frac{\partial p^p}{\partial z} + \frac{\partial}{\partial z}(\eta^p \frac{\partial w^p}{\partial z}) - \phi \frac{\partial p^f}{\partial z} + \phi \frac{\partial}{\partial z}(\eta_{eff} \frac{\partial w^m}{\partial z}) + \frac{F}{1-\phi}(w^m - w^p) - \phi \rho_s g$$

where we have used the fact that $w^f - w^p = \dfrac{w^m - \phi w^p}{\epsilon} - w^p = \dfrac{w^m - w^p}{\epsilon}$

Mixture momentum equation

$$\rho^m \frac{\partial w^m}{\partial t} = -\frac{\partial p^p}{\partial z} - \frac{\partial p^f}{\partial z} + \frac{\partial}{\partial z}(\eta^p \frac{\partial w^p}{\partial z}) + \frac{\partial}{\partial z}(\eta_{eff} \frac{\partial w^m}{\partial z}) - \rho^m g$$

$\Rightarrow$ We now have a system with only w^m, w^p, ϕ, p^f and p^p as variables

Remark 1

Far from the bottom we can assume that:

- the flow is uniform and steady $\longrightarrow \dfrac{\partial w^{m/p}}{\partial z} = \dfrac{\partial}{\partial t} = 0$
- the particle-particle interactions are negligible $\longrightarrow p^p = \overline{\overline{\tau^p}} = 0$

and so the momentum equations simplifies as

$$0 = -\phi \frac{\partial p^f}{\partial z} + \frac{F}{1-\phi}(w^m - w^p) - \phi \rho_s g$$

$$0 = -\frac{\partial p^f}{\partial z} - \rho^m g$$

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$$0 = -\frac{\partial p^f}{\partial z} - \rho^m g$$

Meaning that:

$\Rightarrow$ The fluid pressure gradient balances the weight of the mixture $\rho^m g$

$\Rightarrow$ The drag force balances the apparent weight of the particles

$$0 = \frac{F}{\phi(1-\phi)}(w^m - w^p) - (1-\phi)(\rho_s - \rho_f) g$$

Remark 2

At the bottom, the fluid and the particles are at rest $\longrightarrow w^m = w^p = 0$

and so the momentum equations simplifies as

$$0 = -\frac{\partial p^p}{\partial z} - \phi \frac{\partial p^f}{\partial z} - \phi \rho_s g$$

$$0 = -\frac{\partial p^p}{\partial z} - \frac{\partial p^f}{\partial z} - \rho^m g$$

Remark 2

At the bottom, the fluid and the particles are at rest $\longrightarrow w^m = w^p = 0$

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$$0 = -\frac{\partial p^p}{\partial z} - \phi \frac{\partial p^f}{\partial z} - \phi \rho_s g$$

$$0 = -\frac{\partial p^p}{\partial z} - \frac{\partial p^f}{\partial z} - \rho^m g$$

Meaning that:

$\Rightarrow$ The fluid pressure gradient balances the weight of the fluid $\rho_f g$

$$0 = \frac{\partial p^f}{\partial z} + \rho_f g$$

$\Rightarrow$ The particle pressure gradient balances the apparent weight of the particles $\phi(\rho_s - \rho_f)g$

$$0 = -\frac{\partial p^p}{\partial z} - \phi(\rho_s - \rho_f)g$$

Synthesis

- You know how to derive the **two-phase equations**?

  $\Rightarrow$ Averaging procedure

- You have some ideas about how to **close the two-phase equations** depending on the dilute or dense flow regime?

  $\Rightarrow$ Closure issue

- We have shown that the equation of motion for spherical particles in sedimentation seems to be well represented by the two-phase equations

$\rightarrow$ In the second part of this lecture, we will apply these equations to the case of bed-load transport by laminar shearing flows

[Jackson, 1997] Jackson, R. (1997).
Locally averaged equations of motion for a mixture of identical spherical particles and a newtonian fluid.
Chemical Engineering Science, 52:2457–2469.