Instabilité Hydrodynamique
Hydrodynamic Instability

Activité						Etudiants	Intervenants (accès restreint)	Salles			Equipements	Enseignements	Autres	Autres (accès restreint)
Date	Nom	Semaine	Jour	Heure	Durée	Nom	Nom	Nom	Code	Url	Nom	Nom	Nom	Nom
09/01/2020	Research Courses	s2 06 janv.-12 janv. 2020	Jeudi	08h15	2h		DAVOUST Laurent	G-2B010	Ense3-GreEnER-2B010	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_BE_G3		
09/01/2020	Research Courses	s2 06 janv.-12 janv. 2020	Jeudi	10h30	2h		FLOR Jan-Bert	G-2B010	Ense3-GreEnER-2B010	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_BE_G3		

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19/09/2019	Research Courses	s38 16 sept.-22 sept. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
19/09/2019	Research Courses	s38 16 sept.-22 sept. 2019	Jeudi	13h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
26/09/2019	Research Courses	s39 23 sept.-29 sept. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
26/09/2019	Research Courses	s39 23 sept.-29 sept. 2019	Jeudi	13h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
03/10/2019	Research Courses	s40 30 sept.-06 oct. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
24/10/2019	Research Courses	s43 21 oct.-27 oct. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
14/11/2019	Research Courses	s46 11 nov.-17 nov. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
21/11/2019	Research Courses	s47 18 nov.-24 nov. 2019	Jeudi	13h30	2h		DAVOUST Laurent	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
05/12/2019	Research Courses	s49 02 déc.-08 déc. 2019	Jeudi	10h30	2h		DAVOUST Laurent	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
12/12/2019	Research Courses	s50 09 déc.-15 déc. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
19/12/2019	Research Courses	s51 16 déc.-22 déc. 2019	Jeudi	08h15	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		
19/12/2019	Research Courses	s51 16 déc.-22 déc. 2019	Jeudi	10h30	2h		FLOR Jan-Bert	G-0B007-Amphi Bergès	Ense3-GreEnER-0B007	http://maps.google.fr/maps?q=21 rue des martyrs Grenoble		5EUS5RE_2019_S9_CM_Stability_G1		

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INTRODUCTION to HYDRODYNAMIC INSTABILITY

- 2019 Stability course 26/9/2019
- Literature
- Some observations of instability
- Equations dimensional analyses & simplications
- Concepts for linear stability analyses
- Various flow Examples of calculations.

Instability course 2019

- **Basic approach and equations; normal mode approach.**
  - **Shear flow instability in the presence of density differences**
    Kelvin Helmholtz , Hölmböe, Rayleigh-Taylor, Orr Sommerfeld equations
    Rayleigh and Fjörtöft criterions
  - **Geophysical instabilities: rotational, baroclinic-barotropic, etc,**
  - **Centrifugal instability (Taylor Couette, Görtler, Dean, curved boundaries)**
  - **Convective instabilities**
    Rayleigh Bénard convection (différence de densité)
    Double diffusion (heat and density diffusion)
  - **Capillary instability (jets, Plateau Rayleigh.)**
  - **…**
  - **Interfacial Instabilities (Laurent Davoust)**
  - **Magneto HydroDynamic Instability**
  - **Nonlinear instability**

- *Instabilité Hydrodnamique ; Hydrodynamic instability*
Francois Charru, EDP, 2007; En anglais CUP, 2011.

- *Introduction to Hydrodynamic Stability*
Drazin, P.G. & W.H. Reid, Cambridge University Press.
(1981) and (2000)

- *Hydrodynamic and Hydromagnetic stability*
Chandrasekhar, S. (1961) Dover

Further advanced reading

-*Stability and Transition in Shear Flows,*
P.J. Schmid & D.S. Henningson, Springer, 2001

-*Hydrodynamics and Nonlinear instabilities*, edited by
Godrèche, C. & P. Manneville (1998) Cambridge University
Press, Aléa Saclay collection.

What do we learn from the stability of a flow

1) Is an exact solution stable or unstable ?

2) Very stable … you may find it more often in nature

3) Very unstable…does this flow exists at all, and if it does,
   - under which conditions it is unstable
   - is there a threshold for instability,
   - does the unstable state tends to a stationary state.

Some examples of unstable flows

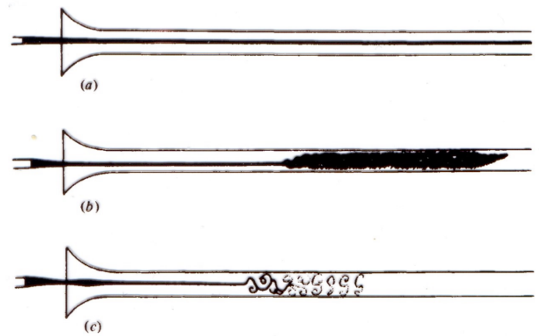
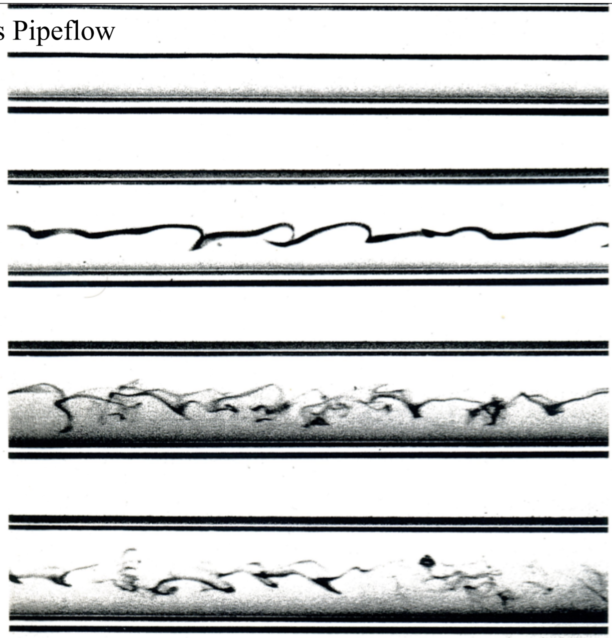


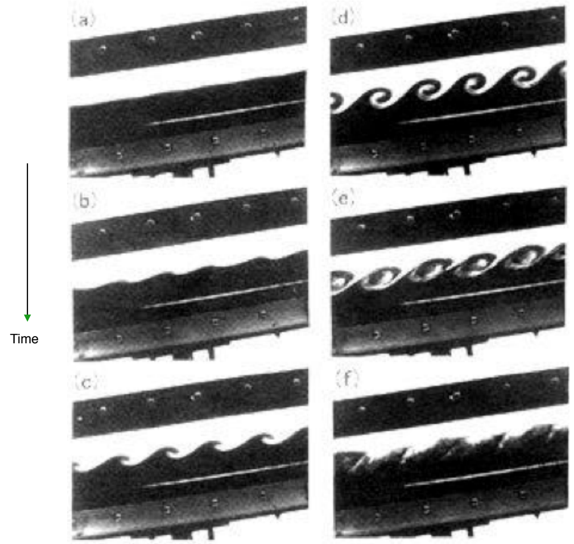
Fig. 1.1. (*a*) Laminar flow in a pipe. (*b*) Transition to turbulent flow in a pipe. (*c*) Transition to turbulent flow as seen when illuminated by a spark. (From Reynolds 1883.)

Fig. 1.2. Turbulent spots in a pipe. (From Reynolds 1883.)

Reynolds Pipeflow



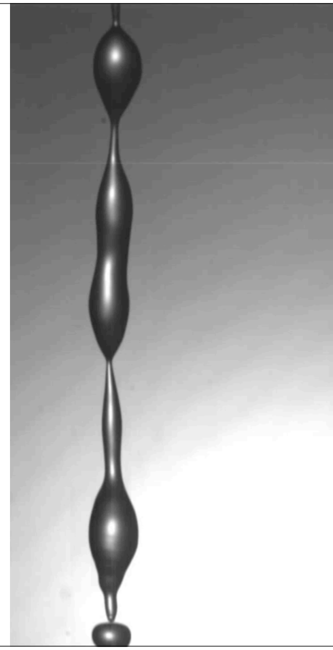
Van Dyke Album of Fluid motion



- Constant wavelength
- Amplitude increase
- reaches a maximum (saturation)
- turbulence

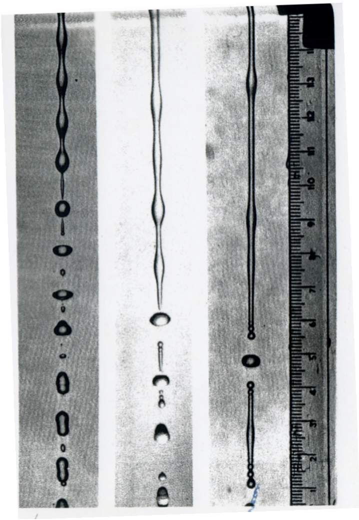
Kelvin Helmholtz (Thorpe 1969)

Plateau-Rayleigh instability



From:BYUSplashLab

http://splashlab.byu.edu

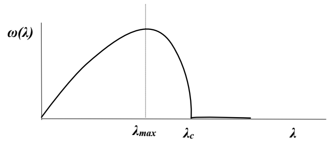


Example (Rayleigh-Plateau)

Pression versus capillary force

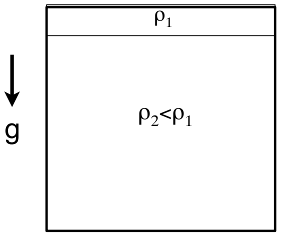
$$p_0 = \sigma \nabla \cdot \mathbf{n} \quad \Rightarrow \quad p_0 = \frac{\sigma}{R_0}$$

The most dangerous wave length λ_{max} appears in the flow and its amplitude grows with ω(λ)



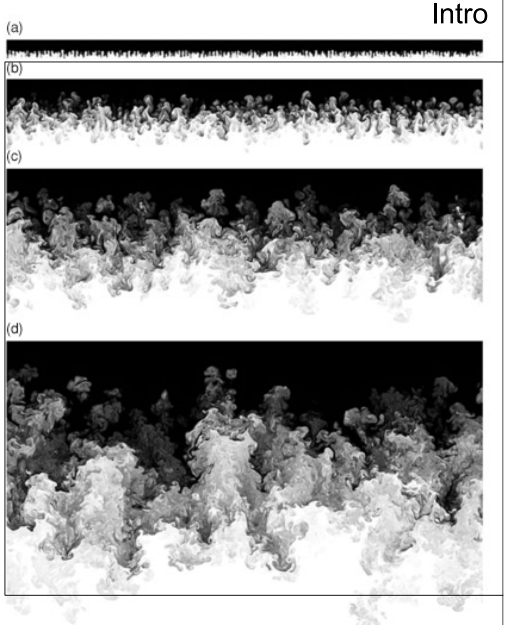
«cutoff wavelength» λ_c ***and 0***

DOUBLE DIFFUSION



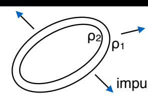
unstable stratification
(salted - fresh water)

instabilité de
Rayleigh Taylor



van Dyke album

Gull 1975: RT in Crab Nebula



Richtmyer Meshkov instability

Instability Mecanism:

Instability: growth of the amplitude of the perturbation of an initially balanced flow

Which balances are there ?

external *forces or internal forces*

External Forces

- Unstable density distributions (under gravity)
- Centrifugal force
- Coriolis force
- Magneto-Hydro-Dynamic Force
- Surface tension

…

Instability Mecanism:

Internal Forces:

- Balance between inertia and pressure force(ν=0)
- In shear flows, instability may depend on vorticity dynamics, vortex line stretching and compression.

  (Viscous effects often stabilise due to the diffusion of momentum; Definition of Reynolds: $Re=UL/\nu \approx$ Inertia/viscosity)

The initial state represents a solution of the equations…

EQUATIONS

Before calculating the stability of a flow, we need to know the basic flow that is perturbed.

Equations are often:	Derived
1) balance of momentum Euler equations Navier Stokes equations	Vorticity equations
2) Conservation of mass, Energy or Volume	Scaled equations Dimensional analyses.
	–> Simplifications

Euler equations :

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\,\nabla \vec{u} = -\frac{1}{\rho}\nabla p - g\vec{k}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{u}) = 0 \ \text{ with } \nabla.\vec{u} = 0 \text{ gives } \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{u}.\nabla \rho = 0$$

no mass diffusion

+ energy equation
+ boundary conditions

NS equations :

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\,\nabla \vec{u} = -\frac{1}{\rho}\nabla p - g\vec{k} + \nu \nabla^2 \vec{u}$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{u}.\nabla \rho = 0$$

+ energy equation
+ boundary conditions

In cartesian coordinates

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In cylindrical coordinates

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$$

$$\Delta = \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In most cases we reduce the equations, using geometric constraints, and/or the dominating force balances.

Balances of forces can be highlighted using scaling arguments (see later dimensional analyses).

Simplifications

Simplifications of equations

- dominating balance between flow forces (as above)
- geometrically confined or limited flows
- in 2D or quasi two dimensional flows e.g. geophysical flows that are confined in one direction (e.g. shallow water -saint venant equations)

Simplifications

Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\,\nabla\vec{u} = -\frac{1}{\rho}\nabla p - g\vec{k}$$

For small density variations $\Delta\rho=\rho_2-\rho_1 \approx 1\%$ we use the Boussinesq approximation, i.e. only density variations in z are considered

The inertia effect on density, i.e. $\Delta\rho\ \partial u/\partial t$ is neglected and only the effect of the gravitational acceleration

$g\,\Delta\rho$ (or often $g' = g\,\Delta\rho/\rho_{mean}$)

on $\Delta\rho$ taken into account.

Hydrostatic balance

If the aspect ratio H/L is small, than we have for horizontal density perturbations, to leading order $\frac{\partial p}{\partial z} = -\rho g$

Bernoulli equation

Suppose a homogeneous, barotropic flow, no density effects, and neglect viscous effect (ν=0) so that we have the Euler equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\,\nabla\vec{u} = -\frac{1}{\rho}\nabla p - g\vec{k}$$

$$\nabla.\vec{u} = 0$$

Introduce a gravitational potential Φ_{gr} with $gk = grad\ \Phi_{gr}$ so that $\Phi_{gr} = -gz$

For the nonlinear term we use the vector identity

$$(\vec{u}.\nabla)\vec{u} = \frac{1}{2}\nabla(\vec{u}.\vec{u}) + (\nabla\times\vec{u})\times\vec{u}$$

and obtain for the Euler equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla(\frac{1}{2}U^2) - \nabla\Phi_{gr} + \frac{1}{\rho}\nabla p = 0$$

vorticity $= \vec{\omega} \equiv \nabla\times\vec{u}$ $\quad U^2 = |\vec{u}.\vec{u}|$ $\quad$ With p=p(ρ) we may write:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{\omega}\times\vec{u} = -\nabla\left(\frac{1}{2}U^2 - \Phi_{gr} + \int\frac{\nabla p}{\rho}\right) = \nabla H} \quad \text{(A)}$$

H is a scalar potential function. We consider a few cases:

1) Steady flow: $\vec{\omega}\times\vec{u} = -\nabla H$

Since $\vec{u}\times(\nabla\times\vec{u}) = \vec{u}\times\vec{\omega} \equiv 0$ along a streamline, we obtain:

$$\frac{1}{2}U^2 + gz + \int\frac{\nabla p}{\rho} = H = \text{constant along streamlines}$$

2) irrotational : $\omega = 0$ we can introduce the velocity potential $\vec{u} = \nabla\phi$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}U^2 + gz + \int\frac{\nabla p}{\rho} = f(t) \text{ in the entire flow field}$$

f(t) is a function of time, and U^2 can be written as $\nabla\phi.\nabla\phi$

3) Steady & irrotational flow:

$$\frac{1}{2}U^2 + gz + \int\frac{\nabla p}{\rho} = H = \text{constant in the entire flow field}$$

equations 3) and (A) above are known as the Bernoulli equation !

4) Steady flow with H= constant:

The Euler equation becomes: $\omega\times\vec{u} = 0$

In 2D flows this implies: $\omega = 0$

In 3D flows ω is parallel to $\vec{u}$ These are known as Beltrami flows.

PERTURBATION OF EQUATIONS AND LINEARIZATION

LAPLACE & FOURIER TRANSFORMS

In linear stability analyses, one supposes a steady basic state U_0 that is perturbed with a perturbation $\mathbf{v}'$, which is restricted to be infinitesimal. The precise meaning of 'infinitesimal' depends on the physical context and the particular experiment.

For instance, one may expand its amplitude A in a Taylor series :

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2 + ...$$

where A_0 is the amplitude of the basic flow, and the small parameter ϵ is a small number that is characteristic for the system under consideration. For instance, when the Reynolds number $Re >> 1$ characterizes the flow, ϵ can be chosen as $\epsilon = 1/Re$.

To know which numbers do characterize the flow we may use, e.g. physical arguments or dimensional analyses.

The perturbation equations are obtained after inserting the time-dependent perturbation. With v of order ϵ we consider leading and first order, i.e. the linear approximation :

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x}, t),$$

$$P(\mathbf{x}, t) = P_0(\mathbf{x}) + p(\mathbf{x}, t)$$

in the equations of motion.

For the Euler equations (viscosity $\nu = 0$), the steady fields must satisfy

$$(\mathbf{U}_0 \cdot \nabla)\mathbf{U}_0 = -\nabla P_0$$

and continuity **?**

$$\nabla \cdot \mathbf{U}_0 = 0.$$

Linearization and making use of the basic state $\mathbf{U}_0$ gives

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{U}_0 \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla) \cdot \mathbf{U}_0 - \nabla p \tag{1}$$

and

$$\nabla \cdot \mathbf{v} = 0$$

with initial conditions $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ and boundary conditions. Subsequently we choose a perturbation amplitude in the form of periodic waves.

(Note that the basic state is subtracted from the equations to obtain the perturbation equations)

Time is eliminated by the Laplace transform of the system with respect to t, seeking solutions of the form

$$\hat{v}_k(r, t) = \hat{v}_k(r) e^{s_k t}$$

where $s_k = s(k) = \omega_R(k) + i\omega_i(k)$ is a complex constant to be determined with the stability analyses ; its value may be different for each different k.
The velocity $\hat{v}$ is to be found from the initial basic velocity field, and the transformed system of ordinary differential equations in r, and the boundary conditions in r.

The perturbation function

The choice of the perturbation function depends on the flow geometry and initial conditions. For a system limited in z and open in x-direction , the perturbation is

$$\sim \hat{v}(z)e^{i(kx+s_k t)}$$

(e.g. Kelvin-Helmholtz $\hat{v}(z)$ is determined with $\nabla^2 v = 0$) ;
For a Poiseuille flow in r-direction and open in x we must analyse perturbations of the form

$$\sim \hat{v}(r)e^{i(s_k t+kx+n\theta)} \text{ and } s_k = \omega_R + i\omega_i$$

Derivatives in x and θ in the equations of motion are transformed into ik and in, repectively, whereas differentiation in the r-direction, ∂_r, leads to an ordinary differential equation in r that needs to be solved with the boundary conditions.

Other boundary geometries

The choice of the symmetry of the disturbance depends on the geometry of the system. For flows with a symmetry axis, for example in the case of a Poiseuille flow, one would take

$$\hat{v}(x,r,\theta,t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{v}_{k,n}(r,t)e^{ikx+in\theta}dk$$

with p, v etc. functions of r. We analyse an arbitrary function in terms of two-dimensional periodic waves with amplitude $\hat{v}(x,r,\theta)$ where $k = \sqrt{k^2+n^2}$ is the wave number associated with the disturbance $\hat{v}_{k,n}$.

Other boundary geometries

For problems with a spherical geometry one would take

$$\hat{v}(x,r,\theta,t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=+l} \hat{v}_l^m(r,t)Y_l^m(\theta,\phi)$$

where $Y_l^m(\theta,\phi)$ represent spherical harmonics. Now, the behaviour of the system with respect to modes l and m has to be investigated. Thus, in all cases the disturbance is expanded in a suitable set of normal modes in accordance with flow geometry.

⟶ Legendre functions

Some concepts

The system of the Hagen-Poiseuille flow $U = V(1 - z^2/d^2)\bar{e}_x$ takes the form of an eigenvalue relation

$$F(s,k,n,V,d,\nu) = 0$$

and eigenfunctions $\hat{v}$, p. (ν is the kinematic viscosity $\nu = \mu/\rho$).
This so-called *method of normal modes* makes use of small disturbances that are resolved into modes which satisfy the linear system and therefore may be treated separately. The use of the Laplace Fourrier transform, thus reduces the equations of motion to an ordinary equation or even an algebraic equation in the parameters of F.

The dispersion relation **Some concepts**

The solution of the ODE (or PDE) + boundary conditions provides the *dispersion relation* for s

$$s = s_n(R, k)$$

where k is the wave number and R the set of control parameters, such as for instance the Reynolds number in the case of the Poiseuille flow.
The fastest growing mode k_c appears the first, and the critical value above which instability occurs.
Because the system is linear, the real and imaginary parts are separate solutions. For stability analyses we are generally interested in the real part of the solutions i.e.

$$\hat{v}_k(r, t) = Re\{\hat{v}_k(r)e^{s_k t}\}$$

without explicitly mentioning it.

The dispersion relation, and stability interpretation **Some concepts**

The growth rate :
(suppose $s = \omega_R + i\omega_i$ and perturbations $\sim e^{st}$ the real part ω_R the exponential growth and the imaginary part ω_i, the sinusoidal part).

for $\omega_R < 0$ the flow is stable
for $\omega_R = 0$ the flow is neutrally stable ,
for $\omega_R > 0$ there is exponential growth.

A flow is marginally stable when $\omega_R = 0$ for critical values on which the eigenvalue ω_R depends, but $\omega_R > 0$ for some neighbouring values of the parameters.
On a neutral curve $\omega_R = 0$, but ω_R is not positive for any of the neighbouring parameters.

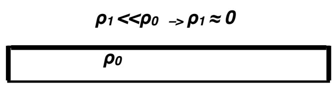
TWO EXAMPLES

Surface waves

Kelvin Helmholtz

EXAMPLE 1

class exercise : **surface waves** perturbation and linearisation:



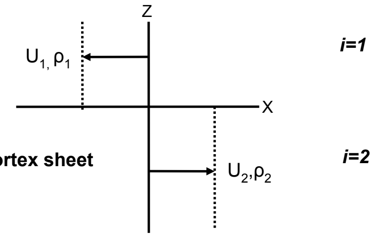
hydrostatic balance

$$\frac{dp_0}{dz} = -\rho g$$

- Consider a basin at rest in hydrostatic balance
  $\boldsymbol{u}'$=(u', w') whereas $p=p_0(x,z)+p'$ and $\rho=\rho_0$.
- Use Bernouilli
  Give the expressions for the leading order O(1) balance and second order O(ε) balance.
- Derive the dispersion relation

EXAMPLE 2

Instability of a vortex sheet



δρ = 0, ρ₁ = ρ₂, Incompressible flow.

$$U_{1,2} = \frac{(U_1 + U_2)}{2} \pm \frac{U_1 - U_2}{2} = C \pm \frac{U}{2}$$

The *frame is moving with speed* C (so that U_i= ±U/2)

Basic flow : vorticity sheet generated by two parallel flows.
The instability is driven by inertial forces.

Velocity potential in each layer $u = \nabla\phi \quad u = \dfrac{\partial\phi}{\partial x} \quad w = \dfrac{\partial\phi}{\partial z}$

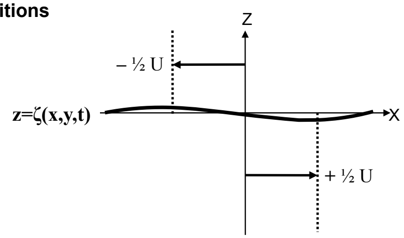
Continuity $\nabla.u = 0$ $\qquad \nabla^2\phi_i = 0$ **Laplace equation**

Note below: upper layer has index *i=1*, lower layer has index *i=2*

Potential flow above and below the interface, we may use *Bernoulli,* $\omega = 0$

$$\frac{\partial\phi_i}{\partial t} + \frac{1}{2}\left(\nabla\phi_i\right)^2 + gz + \frac{P_i}{\rho} = 0$$

Interface conditions



At the interface: the vertical velocity w = vertical interface displacement
particles at interface remain at the interface

$$w_i = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + u_i\frac{\partial\zeta}{\partial x} + w_i\frac{\partial\zeta}{\partial z}$$

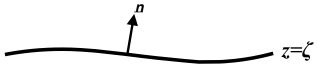
with velocity potential

$$w_i = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi_i}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_i}{\partial z}\frac{\partial\zeta}{\partial z}$$

(Kinematic interface condition)

Forces at the interface are in balance. Here, only pressure, and no tangential forces

$$(P_1 - P_2)_{z=\zeta} = 0$$



With Bernoulli $$\frac{\partial \phi_i}{\partial t} + \frac{1}{2}(\nabla \phi_i)^2 + gz + \frac{P_i}{\rho} = 0$$

$$z = \zeta \qquad \frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\nabla \phi_1)^2 = \frac{\partial \phi_2}{\partial t} + \frac{1}{2}(\nabla \phi_2)^2$$

(Dynamic boundary condition)

The perturbations **Basic flow + perturbation**

above the interface $\phi_1 = \phi_{1b} + \epsilon\phi'_1 + ...$ with $\phi_{1b} = -\frac{1}{2}Ux$

below the interface $\phi_2 = \phi_{2b} + \epsilon\phi'_2 + ...$ with $\phi_{1b} = \frac{1}{2}Ux$

Substitute in the kinematic interface condition

$$w_i = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi_i}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi_i}{\partial z}\frac{\partial \zeta}{\partial z} \qquad \zeta = O(\epsilon)$$

$$w_1 = \frac{\partial \phi'_1}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \left(-\frac{1}{2}U + \frac{\partial \phi'_1}{\partial x}\right)_{z=\zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi'_1}{\partial z}\frac{\partial \zeta}{\partial z}$$

$$w_2 = \frac{\partial \phi'_2}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \left(\frac{1}{2}U + \frac{\partial \phi'_2}{\partial x}\right)_{z=\zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi'_2}{\partial z}\frac{\partial \zeta}{\partial z}$$

Linear approximation : **keep O(ε) terms, neglect O(ε²) terms:**

$$w_1 = \frac{\partial \phi'_1}{\partial z} = \frac{\partial \zeta}{\partial t} - \frac{1}{2}U\frac{\partial \zeta}{\partial x}$$

$$w_2 = \frac{\partial \phi'_2}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{1}{2}U\frac{\partial \zeta}{\partial x}$$

In dynamic boundary condition

$$z = \zeta \qquad \frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\nabla \phi_1)^2 = \frac{\partial \phi_2}{\partial t} + \frac{1}{2}(\nabla \phi_2)^2$$

with $\nabla\phi_1 = -\frac{1}{2}U + \frac{\partial \phi'_1}{\partial x}$ and $\nabla\phi_2 = \frac{1}{2}U + \frac{\partial \phi'_2}{\partial x}$

$$\left(\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1}{\partial t}\right)_{z=\zeta} = \frac{U}{2}\left(\frac{\partial \phi'_2}{\partial t} + \frac{\partial \phi'_1}{\partial t}\right)_{z=\zeta}$$

From the three conditions we have:

Laplace equation $\nabla^2 \phi_i = 0$

Kinematic BC

$$w_1 = \frac{\partial \phi'_1}{\partial z} = \frac{\partial \zeta}{\partial t} - \frac{1}{2}U\frac{\partial \zeta}{\partial x}$$

$$w_2 = \frac{\partial \phi'_2}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{1}{2}U\frac{\partial \zeta}{\partial x}$$

Dynamic BC $$\left(\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1}{\partial t}\right)_{z=\zeta} = \frac{U}{2}\left(\frac{\partial \phi'_2}{\partial t} + \frac{\partial \phi'_1}{\partial t}\right)_{z=\zeta}$$

perturbations:

$$\phi_i = F(z)\exp(ikx + \sigma t) \text{ and } \zeta = A\exp(ikx + \sigma t)$$

With ***F(z)*** the vertical dependence to determine, and ***A*** the amplitude

The wave form is sinusoidal, with spacing $\lambda = 2\pi/k$

With the Laplace transform, $e^{\sigma t}$ exponential decay or growth is supposed

$$\nabla^2\phi_i = 0 \quad \frac{d^2\phi_i}{dx^2} + \frac{d^2\phi_i}{dz^2} = 0$$

$$\frac{d^2F}{dz^2} - k^2F = 0$$

Condition at infinity: the amplitude of the perturbations goes to zero

$$\phi_i = B_1e^{-kz} + B_2e^{kz}$$

$$\phi_i \to 0 \qquad z \to +\infty \text{ thus for } z > 0 \;\; B_2 = 0$$

$$\phi_i \to 0 \qquad z \to -\infty \text{ thus for } z < 0 \;\; B_1 = 0$$

We can now **solve the form of ζ^*, φ^*_1, φ^*_2 with amplitudes A, B_1, and B_2**

$\zeta = \mathbf{A}\, e^{ikx + \sigma t}$,

$\varphi'_1 = B_1\, e^{-kz}\, e^{ikx + \sigma t}$. $\qquad \varphi'_2 = B_2\, e^{kz}\, e^{ikx + \sigma t}$

Substitution in conditions I and II:

$-kB_1 = (\sigma - \tfrac{1}{2}i\,k\,U)\,A$

$-kB_2 = (\sigma + \tfrac{1}{2}i\,k\,U)\,A$

and condition III: $\quad i\,[\sigma(B_2 - B_1)_{z=0} + \tfrac{1}{2}U\,(B_2k + B_1k)_{z=0}]\,e^{i(kx)} = 0$

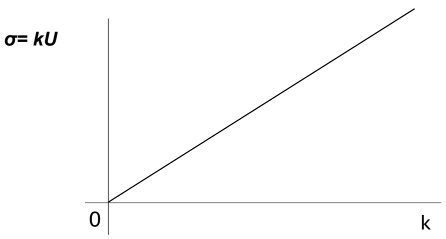
$$\boldsymbol{\sigma = \tfrac{1}{2}ik(U_1+U_2) \pm \tfrac{1}{2}k(U_1 - U_2)}$$

for $\mathbf{U_1 = -U_2}$ this reduces to

$$\boldsymbol{\sigma = \pm kU}$$

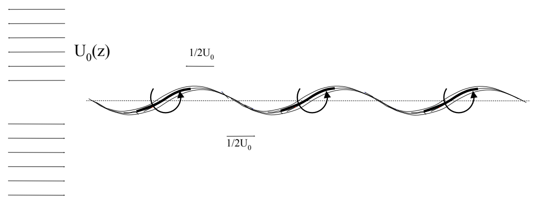
σ(k) is the ***dispersion relation*** showing the variation of growth rate with k.
For σ>0, k≠0 the sheet is unstable. Small wavelengths grow faster than short ones.

- exponential growth for any velocity for σ>0
- growth rate depends on U



$\sigma = \pm kU$

All wave lengths are unstable no matter how small U is!.
In reality often there is a cutoff for small wavelengths
as we will see later.



(Vertical velocity is out phase with pressure and velocity, and horizontal vorticity $\omega_y \approx -\frac{\partial w}{\partial x} = -ikw$)

SCALING

Scaling

Buckingham theorem: find dimensionless numbers

Consider the non-dimensional parameter a to determine, that depends on n governing parameters

$$a = f(a_1, a_2, ..., a_k, b1, ..., b_m)$$

with k independent parameters a_i, and m dependent parameters b_i, $m + k = n$ and f is a function. Since b_i are dependent parameters we can write them as a function of the independent parameters a_i :

$$b_1 = [a_1]^{p_1}...[a_k]^{r_1}$$

$$b_m = [a_1]^{p_m}...[a_k]^{r_m}$$

Scaling

For this Poisseuille flow the parameter a to determine is the pressure gradient

$$\frac{dp}{dx} = f(U, d, \rho, \mu)$$

we have the dimensions :

$$[\frac{dp}{dx}] = ML^{-2}T^{-2}$$
$$[U] = LT^{-1}$$
$$[D] = L$$
$$[\rho] = ML^{-3}$$
$$[\mu] = ML^{-1}T^{-1}$$

In this case U, D, and ρ are independent ($[\mu] = [\rho][u][D]$), so that $k = 3$, $m = 1$ and $n = 4$ and so $\Pi = \Phi(\Pi_1)$

Scaling

n=4 governing parameters with V, D, ρ independent and one dependent parameters μ. So

$$\Pi_1 = \frac{\mu}{UD\rho}$$

(this is 1/Re) and Π the dimensionless pressure gradient

$$\Pi = \frac{1}{(U^2D^{-1}\rho)}\frac{dp}{dx}$$

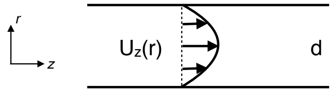
We thus obtain :

$$\Pi = (U^2D^{-1}\rho)^{-1}\frac{dp}{dx} = \Phi(\Pi_1) = \Phi(1/Re)$$

and Φ the function to determine. Whatever the individual values of d, U, μ or ρ are, this function is universal. Here Φ is determined experimentally, see picture.

Scaling and Dimensionless equations

Consider the case of a steady laminar flow in a tube



$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\,\nabla \vec{u} = -\frac{1}{\rho}\nabla p - g\vec{k} + \nu \nabla^2 \vec{u}$$

$u_r = u_\theta = 0$, axisymmetric flow, and developed flow, i.e. $\partial u_z/\partial z = 0$, so that $\partial u/\partial t = 0$, *and no slip condition at the walls.*

Pressure -viscous force balance:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0$$

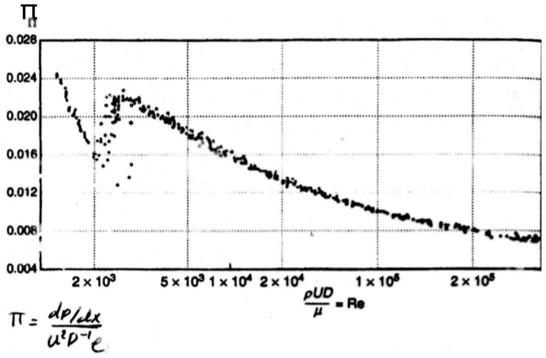
We have length scale d, velocity scale U_0 –> pressure $P/\rho \sim U^2$

$$-\frac{\partial p'}{\partial z} + \frac{1}{Re}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0$$

with p' u, z and r dimensionless

Flow solution $u_z = -Re\frac{\partial p}{\partial z}(1-(r/d)^2)$ amplitude~ pressure drop and Re

Scaling



For stability of Poisseuille flow see e.g. Drazin & Reid.

Role of viscosity and generation of vorticity in the boundary layer play an important role (discussed in shear layer instability).

Similarity analyses, Barenblatt 1996 : Scaling, self-similarity, and intermediate asymptotics, CUP

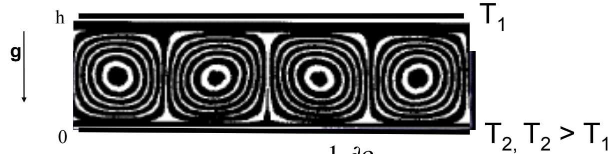
Methods :

- Normal mode analyses
  Gives information about instability growth rate, and corresponding wave lengths in the linear approach.

- Energy balance of potential and/or kinetic Energy;
  Considering the motion of particles.

Intro

Heuristic scaling in convection



Thermal expansion coefficient $\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$

Force/Volume = ρ . acceleration = g ρ α ΔT
suppose that the particle accelerates during a time τ_A, going from z=0 to h

Diffusion opposes the motion due to:
- Vorticity diffusion: the velocity gradient is damped by viscosity ν
- Temperature gradient diffusion with
  Thermal conductivity κ=X/C and X heat conductivity and C the capacity

Dimensions of diffusion are: $\frac{\partial}{\partial t}(...) = \nabla^2(...) \sim l^2/t$ --->

The characteristic times of diffusion are:

$\tau_\nu = h^2/\nu$ … for vorticity diffusion
$\tau_\theta = h^2/\kappa$ … for heat diffusion

There is a competition between the acceleration (τ_A) opposed by diffusion effects. The ratio in time scales determines stability:

$$Ra = \tau_\nu \tau_\theta / \tau_A^2 = \frac{\alpha g \Delta T h^3}{\kappa \nu}$$

(Rayleigh number)

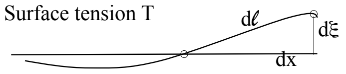
Ra>1 convection

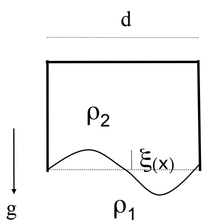
Ra<1 stable… in reality Ra>673 or higher

(but the number is right)

Interfacial instability

Balancing forces:
- Surface tension *minimizes* the surface
- Gravity *increases* the surface





Consider the flow energy

Surface-tension (vertical): $T(d\ell - dx) = T\,[\,(dx^2 + d\xi^2)^{1/2} - dx]$
$= T\{[1 + (d\xi/dx)^2\,]^{1/2} - 1\}dx \approx ½\, T(d\xi/dx)^2$

$d\xi/dx \ll 1$

Gravity potential energy: $(\rho_1 - \rho_2) g dz = \frac{1}{2}(\rho_1 - \rho_2) g \xi(x) dx$

Energy ΔE= $\int_0^d \frac{g}{2}(\rho_1 - \rho_2)\xi(x) + \frac{T}{2}\left(\frac{d\xi}{dx}\right)^2 dx$ <0 unstable and for >0 stable

Interfacial instability Normal mode decomposition

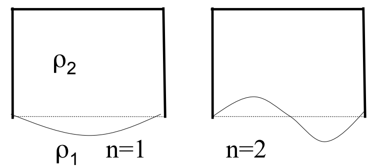
Discrete number of modes:

$$\xi = \sum_k \xi_k e^{ikx} \qquad \Delta E = \sum_k |\xi_k| \left[\frac{1}{2}(\rho_1 - \rho_2) g + \frac{T}{2} k^2\right]$$

with $k = n\pi/d$ $\qquad \xi \propto \sin \frac{n\pi x}{d}$

Instability when $-(\rho_2 - \rho_1)g + Tk^2 < 0 \qquad \Delta\rho = \rho_2 - \rho_1$

That is the case when $k^2 < \Delta\rho g/T \longrightarrow n\pi/d < (\Delta\rho g/T)^{1/2}$



take n=2 : λ?
$2\pi/\lambda < (\Delta\rho g/T)^{1/2}$ for instability
$\Delta\rho = 0.2\,g/cm^3$, $T = 70.10^{-3}\,N/m$
$\lambda = (\Delta\rho g/T)^{-1/2} = 1.6\,cm$

Home exercise: A NORMAL MODE TOY PROBLEM

Suppose there is a balance represented by the 1D equation:

$$\frac{\partial f(y)}{\partial t} = f(y) - f(y)^2 + \frac{1}{\lambda}\frac{\partial^2 f(y)}{\partial y^2}$$

with basic state *f=0,* and boundary conditions *f(0)=f(1)=0.*

1. Is the basic state a solution of the equation ?
2. Perturb the basic flow by adding a perturbation *f'*

$$f(x,t) = \bar{f} + \epsilon f'(x,t) + \epsilon^2 ...$$

3. *S*ubstitute, separate O(0), O(ε), …and consider order ε only
4. What happens at O(0) ? Consider the equation for O(ε).
5. Use the Laplace transformation *f'(t)= F(y)* $e^{\sigma(k)t}$ and solve *F(y)* with the boundary conditions.
6. What is the dispersion relation and what does this relation show ?