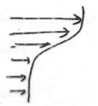
# Instability of Parallel Shear Flows

## Free shear flows

**free shear layer**
with $U=U_0 \tanh(y/L)$



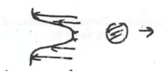
**unstable for**

$Re_c=0$

**jets** (Bickley jet)
with $U=U_0 \,\mathrm{sech}^2(y/L)$

$Re_c=4$

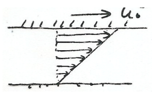
**wakes**



**heated wall**
(free convection)

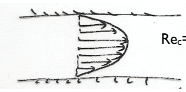
## Bounded shear flows

**plane Couette flow**
with $U=U_0\, y/L$




$Re_c=\infty$

**plane Poiseuille flow**
with $U=U_0\,[1-(2y/L)^2]$

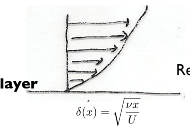


$Re_c=540$

**Poiseuille pipe flow**
with $U=U_0\,[1-(r/R)^2]$

$Re_c=\infty$

**Blasius boundary layer**




$Re_c=520$

Refs.: Drazin & Reid 2004, Ch4p124-165; Godreche & Manneville 1998, Ch2.; Schlichting& Gersten 2000

# Reynolds stress

The Reynolds stress is a mechanism of energy (or momentum) transfer from the mean flow to the perturbations. This stress is able to sustain or amplify the perturbations.

Two criteria can be explained with the transfer of momentum by the Reynolds stress:

- The *Rayleigh inflection-point,* and the *Fjörtöft* criteria.

( *Viscous effects* generally damp the perturbations…
But near the boundary, they may destabilise the flow.)

# Reynolds stress

The Navier-Stokes perturbation equations are:

$$\frac{\partial(U_i+u_i)}{\partial t}+(U_j+u_j)\frac{\partial(U_i+u_i)}{\partial x_j}=-\frac{1}{\rho}\frac{\partial(P+p)}{\partial x_i}+\nu\frac{\partial^2(U_i+u_i)}{\partial x_j^2}$$

Average in time $\quad \overline{\frac{\partial u}{\partial x_i}}=\frac{\partial \bar{u}}{\partial x_i},\quad \bar{u}_i=0$

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}+\overline{u_j\frac{\partial u_i}{\partial x_j}}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j^2}$$

Suppose that the flow is steady (i.e. $\partial/\partial t=0$), with continuity, i.e. $\partial u_i/\partial x_i=0$, and 2D flow we obtain

$$U_j\frac{\partial U_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j^2}-\frac{\partial \overline{u_i u_j}}{\partial x_j}\quad \underset{2D}{\Longrightarrow}\quad U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 U}{\partial y^2}-\frac{\partial \overline{uv}}{\partial y}$$

$$\boxed{U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}-\rho\overline{uv}\right)}$$

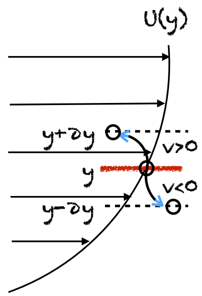
The quantity $-\rho\overline{uv}$ (in general $-\rho\overline{u_i u_j}$ ) with $u$ and $v$ the velocity perturbations, is called the Reynolds Stress. This term is responsible for the transport of momentum from the mean flow to the perturbations.

# Reynolds stress

The Blasius boundary layer flow

Stress exerted by the turbulent fluctuations on the mean flow; these fluctuations transport mean momentum, and in doing so, may render the mean flow unstable

Consider a particle in a shear flow




v and u are the perturbation velocities

for v > 0

at the position y+ δy the mean velocity is U(y+ δy) >U(y) so that the particle moves relatively slower than its ambient. The perturbation to the horizontal velocity U, is u<0, therefore the correlation $\overline{uv}<0$

for v < 0

at the position y – δy the mean velocity is U(y – δy) <U(y) the particle moves relatively faster than its ambient u>0, again the correlation $\overline{uv}<0$

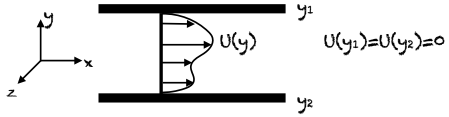
on average $\overline{uv}<0$ so that the term $\frac{\partial}{\partial y}(-\rho\overline{uv}) >0$

The turbulence transfers momentum and in doing so changes dU/dy

**This may be stabilising or destabilising !**

## Reynolds stress (Rayleigh's and Fjörtöft's criteria)

Consider an inviscid (incompressible) parallel flow with $U = U(y)$ and $P_0(x,t) = P_0$, $\rho$=constant, $\vec{u} = (u, v, w)$.




The dimensionless Euler equations :

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U}.\nabla)\vec{U} = -\nabla P$$
$$\nabla.\vec{U} = 0$$

Perturbations have the form

$$\vec{U}(x,t) = U(y)\vec{e}_x + \vec{u}(\vec{x},t)$$
$$P(\vec{x},t) = P_0 + p(\vec{x},t)$$

so that we obtain for the perturbation equation (as usual)

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\vec{u} + v\frac{\partial U}{\partial y}\vec{e}_x = -\nabla P$$
$$\nabla.\vec{u} = 0$$

The perturbation equations are :

$$\begin{aligned} ik_x + ik_z + \frac{d\hat{v}}{dy} &= 0 \\ ik_x\,[U(y) - c]\,\hat{u} + \frac{dU}{dy}\hat{v} &= ik_x\hat{p} \\ ik_x\,[U(y) - c]\,\hat{v} &= \frac{dp}{dy} \\ ik_x\,[U(y) - c]\,\hat{w} &= ik_z\hat{p} \end{aligned}$$

and $c = \omega/k_x$ is the *complex phase velocity*.
Boundary conditions are $\hat{v}(y_1) = 0$ and $\hat{v}(y_2) = 0$.

Find solutions of the *dispersion relation* $D(k,\omega) = 0$ in 3D !

Use Squires theorem , and transform the 3D stability problem into the equivalent 2D problem $\rightarrow$

## Intermezzo

Squires theorem : (see Drazin & Reid 1981 ; Godreche et Manneville 1998).
"To obtain the minimal critical Reynolds number for instability, it is sufficient to consider only two dimensional perturbations."

Use the appropriate transformation of variables.

$$\tilde{k}^2 = k_x^2 + k_z^2$$
$$\tilde{k}\tilde{u} = k_x\hat{u} + k_z\hat{w}$$
$$\tilde{v} = \hat{v}$$
$$\tilde{p}/\tilde{k} = p/k_x$$

so that the perturbation equations become

$$\begin{aligned} i\tilde{k}\tilde{u} + \frac{d\tilde{v}}{dy} &= 0 \\ i\tilde{k}[U - \tilde{c}]\tilde{u} + \frac{dU}{dy} &= -ik\tilde{p} \\ i\tilde{k}[U - \tilde{c}]\tilde{v} &= -\frac{d\tilde{p}}{dy} \\ \tilde{v}(y_1) = \tilde{v}(y_2) &= 0 \end{aligned}$$

## Intermezzo

For the 2D case, the dispersion relation is

$$\tilde{D}(\tilde{k},\tilde{\omega}) = 0$$

with wavenumber $\tilde{k}$ and $\tilde{\omega} = \tilde{k}\frac{\tilde{\omega}}{k_x} = \frac{(k_x^2+k_z^2)^{1/2}}{k_x}\omega$, so that the 3D dispersion relation is :

$$D(\vec{k},\omega) \equiv \tilde{D}\left[(k_x^2 + k_y^2)^{1/2}, \frac{(k_x^2 + k_z^2)^{1/2}}{k_x}\omega\right] = 0$$

From the 2D relation for $(\tilde{k},\tilde{\omega})$ we can obtain the properties of the 3D waves $(\vec{k},\omega)$. For the growth rate, $\omega_i$, we note that

$$\frac{(k_x^2 + k_z^2)^{1/2}}{k_x}\omega_i > \omega_i$$

and thus always $\omega_i(2D) > \omega_i(3D)$.
For stability we can thus consider the 2D problem !

The 2D stability problem ($\nu = 0$).
Introduce the streamfunction $\Phi$
with $u = \frac{\partial \Phi}{\partial y}$ and $v = -\frac{\partial \Phi}{\partial x}$ and vorticity $\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \Phi$.

Conservation of vorticity implies :

$$\frac{D\Omega}{Dt} = \left[\frac{\partial}{\partial t} + u.\nabla)\right]\Omega = \left[\frac{\partial}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}\right]\nabla^2\Phi = 0$$

Perturbations : are of the form $\Phi(x,y,t) = \int U(y)dy + \psi(x,y,t)$

$$\left[\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right]\nabla^2\psi - \frac{d^2U}{dy^2}\frac{\partial \psi}{\partial x} = 0$$

Normal modes :

$$\begin{aligned} \phi &= Re\{\phi(y)e^{i(kx-\omega t)}\} \\ u &= Re\{\frac{d\phi(y)}{dy}e^{i(kx-\omega t)}\} \\ v &= -Re\{ik\phi(y)e^{i(kx-\omega t)}\} \end{aligned}$$

Boundary condition : $\phi(y_1) = \phi(y_2) = 0$ below $e^{ik(x-Ct)}$

... we obtain Rayleighs equation $(U-c)\left[\phi'' - k^2\phi\right] - U''\phi = 0$
If we suppose $U \neq c$, then

$$\left[\phi'' - k^2\phi\right] - \frac{U''}{(U-c)}\phi = 0 \qquad (1)$$

If $\phi$ is a solution, then so is its complex conjugate.
Integration of (1) from $y_1$ to $y_2$ gives

$$\int_{y_1}^{y_2}(|\phi'|^2 + k^2|\phi|^2)dy + \int_{y_1}^{y_2}\frac{U''}{(U-c)}|\phi|^2 dy = 0 \qquad (2)$$

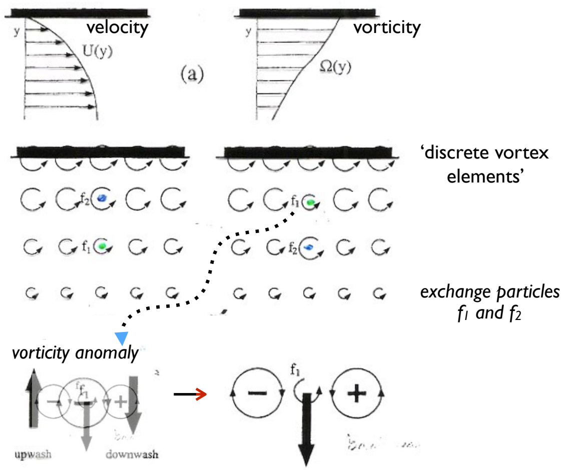
$$\frac{U''}{U-c}|\phi|^2 = \frac{U''(U-c^*)}{|U-c|^2}|\phi|^2 = \frac{U''(U-c_r+ic_i)}{|U-c|^2}|\phi|^2$$

so that we can write for the imaginary part

$$c_i\int_{y_1}^{y_2}\frac{U''}{|U-c|^2}|\phi|^2 dy = 0$$

either stable flow ($c_i = 0$), or instability ($c_i \neq 0$) and $\int_{y_1}^{y_2}..dy = 0$.
For instability there must be an inflection point $U''(y) = \frac{d^2U}{dy^2} = 0$.

# Explanation for stability




$$[\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}]\omega_z - \frac{\partial \Omega}{\partial y}v = 0$$

from Godreche et Manneville, 1989

Fjörtöft stability criterion. (From energy conservation.)
Derive the energy equation from the Euler's equations :
$[\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}](\frac{1}{2}(u^2+v^2+w^2)) + \nabla.[p + \frac{1}{2}(u^2+v^2+w^2)]\mathbf{u}$

$$= -U'(y)uv - U'(y)U(y)v$$

After taking the spatial average, i.e. integrate in wave space (x,y) from (0,0) to $(\lambda_x, \lambda_y)$ to give

$$\frac{\partial}{\partial t}(\frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})) + \frac{\partial}{\partial y}.\overline{[p + \frac{1}{2}(u^2+v^2+w^2)v]} = -U'(y)\overline{uv}$$

With boundary conditions $v(y_1) = v(y_2) = 0$ this can be further reduced to

$$\frac{\partial}{\partial t}\int_{y_1}^{y_2}\left[(\frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}))\right]dy = -\int_{y_1}^{y_2} -U'(y)\overline{uv}dy$$

This latter relation shows the kinetic energy perturbation in response to the work done by Reynolds stress.

Fjörtöft stability criterion. Rewrite equation (2) in the form

$$\int_{y_1}^{y_2} (|\phi'|^2 + k^2|\phi|^2)dy = -\int_{y_1}^{y_2} \frac{U''(U-c_r)}{|U-c|^2}|\phi|^2 dy$$

assume Rayleigh's criterion
and add (with $U_s$ the velocity at the inflection point)

$$(c_r - U_s)\int_{y_1}^{y_2} \frac{U''}{|U-c|^2}|\phi|^2 dy = 0$$

so that we obtain

$$\int_{y_1}^{y_2} \frac{U''(U-U_s)}{|U-c|^2}|\phi|^2 dy = -\int_{y_1}^{y_2} (|\phi'|^2 + k^2|\phi|^2)dy < 0$$

Since the rhs is always negative, this implies for instability that

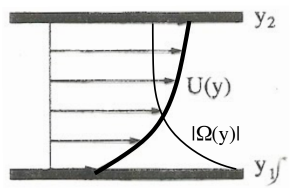
$$U''(U-U_s)0 \le 0 \text{ in the domain} y_1 \le y \le y_2$$

For a monotonic velocity $U$, the absolute value of the vorticity $|\Omega(y)| \equiv |\frac{d^2U(y)}{dy^2}|$ has then a maximum at the inflection point $y_s$.

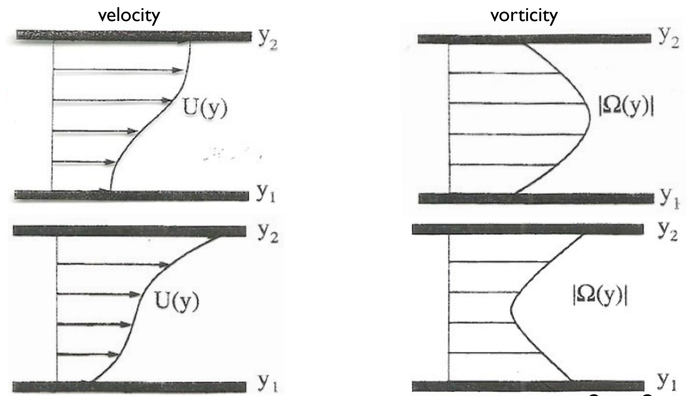
# Notes

- The mechanism to explain the Rayleigh inflection point theorem of shear flows, does not apply to the Fjörtöft criterion (see Orszag & Patera 1981).
- one can derive the Rayleigh criterion from conservation of momentum (see Bayly J. Fluid Mech. 1988, and Godreche & Manneville 1998).
  This approach shows that the Reynolds stress term plays an important role for the mechanisms of shear instability.
- The inflection point theorem works equally for bounded and unbounded flows.
- A similar approach is possible also for the Fjörtöft criterion, based on energy conservation (viscous effects are neglected).
- These are *NECESSARY* conditions for instability but *NOT SUFFICIENT* :
  - not all flows with inflexion point are unstable, but if the flow is unstable, then it must have an inflection point.

  Counter examples are:
  - *U=sin (z)* with inflexion points at $z_s=n\pi$ but stable flow (check Fjortoft and Rayleigh's criterion)
  - Rayleigh's shear flow, i.e. $U= z/b$ (for $|z|<b$) with $|z|\le 1$
    $U= -1$ for $z< -b$ ; $U=1$ for $z<b$
    No inflection point, but this flow is unstable (see course 2)




Rayleigh: $U''(z) = 0$
Fjortoft: $U''(U-U_s) \le 0$ and $U(z) = U_s$




Which flows are unstable ? And in case ∂ρ/∂z≠0 ?

Flow on curved Boundaries

Favorable and adverse pressure gradients.

Flows over any non-flat surface may provoke an adverse or favourable pressure gradient. If there is a return flow it may generate an inflection point, so that Rayleigh's and Fjörtöft criteria apply.

The boundary layer equation is :

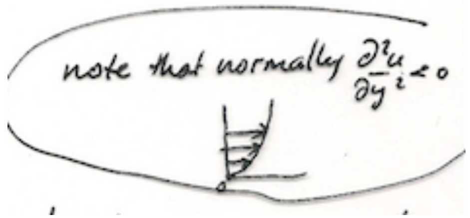
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

The pressure is determined by the external velocity field

$$\boxed{\frac{\partial p}{\partial x} = -\rho U\frac{\partial U}{\partial x}}$$

where $x$ is along the surface of the body. At the wall we have

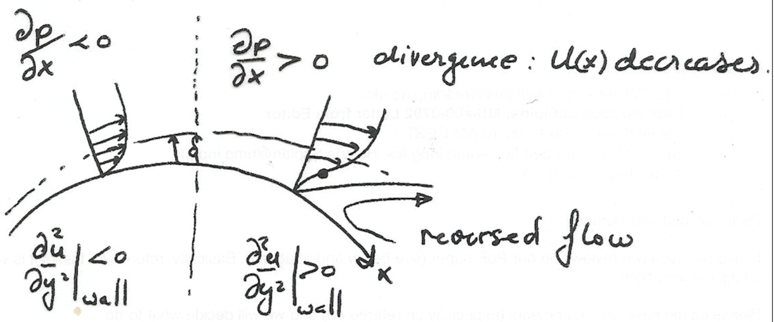
$$\mu\frac{\partial^2 u}{\partial y^2}|_{wall} = \frac{\partial p}{\partial x}$$



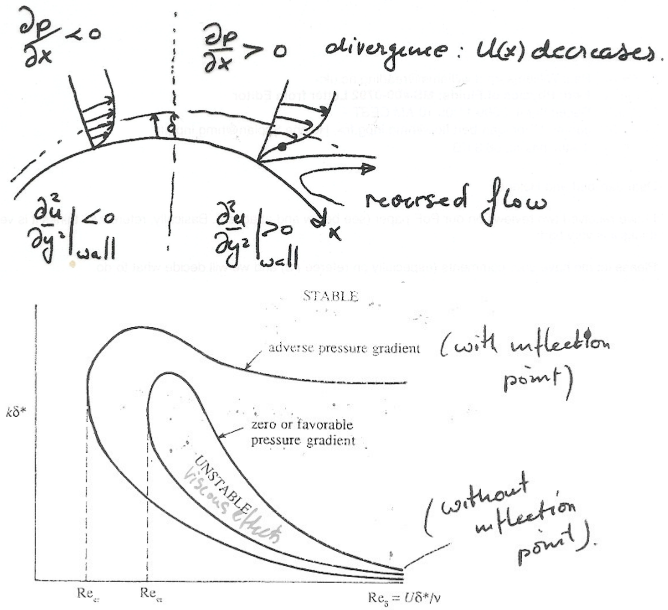
Accelerating flow : $\frac{\partial p}{\partial x} < 0 \rightarrow \frac{\partial^2 u}{\partial y^2} < 0$

Decelerating flow : $\frac{\partial p}{\partial x} > 0 \rightarrow \frac{\partial^2 u}{\partial y^2} > 0.$

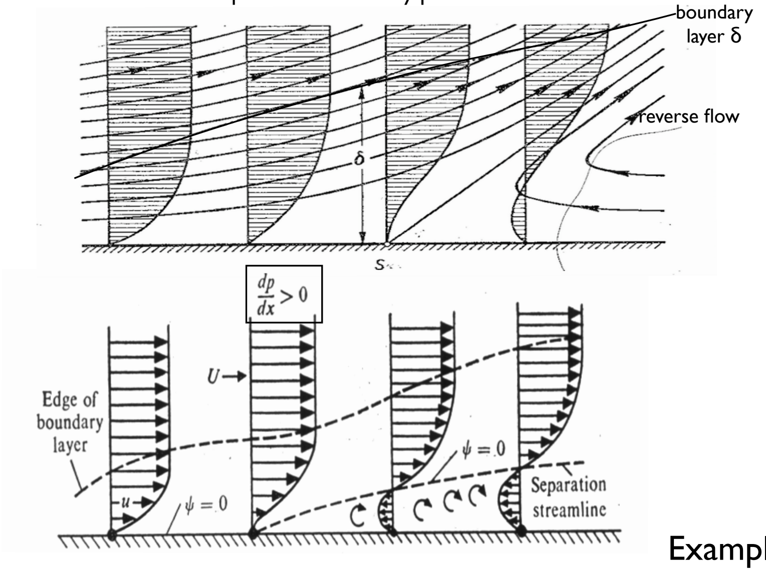
This means that somewhere in the boundary layer $\frac{\partial^2 U}{\partial y^2} = 0$, i.e. there is an inflection point.



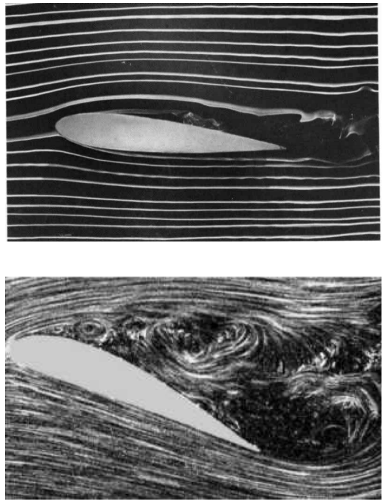

Reversed flow --> separation point is defined as $\frac{\partial u}{\partial y}|_{wall} = 0$



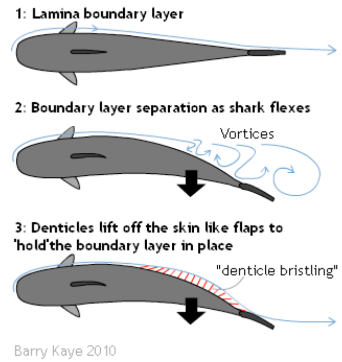
Boundary layer flows over obstacles show reverse flows that lead to an inflection point in the velocity profile




Example?



# Sharks control the BL




# Critical layers

Drazin & Reid p 133.

In Rayleigh's equation

$$(U - c)\,[\phi'' - \alpha\phi] - U''\phi = 0 \qquad c = U_c \quad at \quad y = y_c$$

*note that* $\sim \phi(y)e^{i(kx-\omega t)}$ *and* $\phi' = \frac{\partial \phi}{\partial y}$

Solutions found by Tollmien (1929) and others (see Drazin & Reid, Godreche & Manneville 1998 ...)

$$[\phi'' - \alpha\phi] - \frac{U''}{(U - U_c)}\phi = 0 \qquad c = U_c \quad at \quad y = y_c$$

This is a singular point. The region around is called the critical layer. Solutions of the Rayleigh equation are now given by

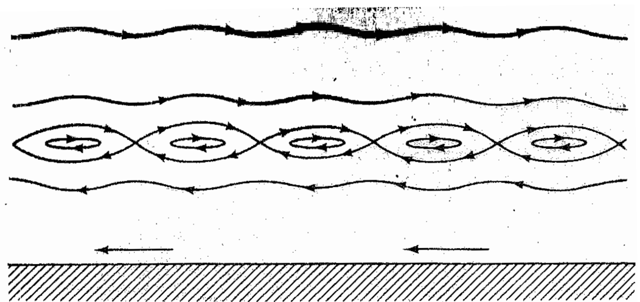
$$U - c = U_c'(y - y_c)$$

and after substitution and integration

$$u' = \phi' \approx \frac{U''}{U_c'} ln(y - y_c)$$

If $U''$ does not go to zero at $y{=}y_c$, then $\phi' \longrightarrow \infty$. This means that there is a singular layer of vorticity at $y{=}y_c$.
This singularity can be smoothed out by viscosity, and if not we may see —>

the so-called 'cat's eye' pattern of vortices



Kelvin cat's eye pattern of the streamlines near the critical level as viewed by an observer moving with the wave. (Drazin & Reid p141)

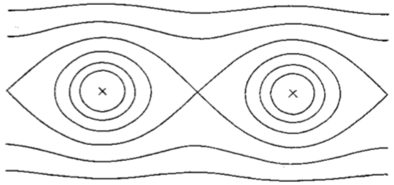
The stream function is ($A$ is the amplitude of the wave)

$$\int_y^y (U - c)dy + A\,Re\left\{\phi(y)e^{i\alpha x}\right\}$$

so that near the level $y_c$

$$\frac{1}{2}U_c'(y - y_c)^2 dy + A\,\phi(y_c)\cos\alpha x = \ constant$$

# vortex street, row of line vortices



This layer can be represented by a row of line equidistant vortices of equal strength (see Lamb 1932 art.156), distance $d$ and strength $A$.
The wave number can be represented by $k=2\pi/d$.

In the complex plane $z=x+iy$ and $w=\phi+i\psi$ the potential and streamfunction.

$$w = \phi + i\psi = \frac{iA}{2\pi}\log\{\sin\frac{\pi}{d}z\} = \frac{iA}{2\pi}\log\{\sin[k(x+iy)/2]\}$$

so that the velocity is $$u - iv = \frac{dw}{dz} = -\frac{4i\,A\,k}{\pi}\cot[k(x+iy)/2]$$

for large z $$u = -\frac{4kA}{\pi}\frac{\sinh ky}{\cosh ky - \cos kx} \qquad v = \frac{4kA}{\pi}\frac{\sin ky}{\cosh ky - \cos kx}$$

Flow is unstable (Lamb 1932)

# Derive :

The Taylor-Goldstein equation

- Parallel flow U(z) $[U + u', v', w']$ and stratification N.
- Euler Equations, viscosity $\nu = 0$
- Squires theorem $(\nu = 0)$ : $3D \to 2D$, stronger growth for 2D than 3D (see later)
- linearize, define a stream function $u' = \frac{\partial\psi}{\partial z}$ $\quad w' = -\frac{\partial\psi}{\partial x}$
- perturbation $[\rho, p, \psi] = [\hat{\rho}(z), \hat{p}(z), \hat{\phi}(z)]e^{[ik(x-ct)]}$

$\rightarrow$

$$(U-c)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{N^2}{(U-c)} - U_{zz}\right\}\phi = 0$$

# Parallel flow over a boundary, viscous effects

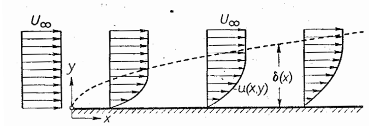



Fig. 2.2. Sketch of boundary layer on a flat plate in parallel flow at zero incidence

$\delta$ boundary layer thickness due to the diffusion of vorticity from the boundary (selfsimilar solutions, see Batchelor 1969).

For a velocity $U$, layer thickness $\delta$ (in y-), and plate length $l$ (in x-direction) we can estimate the terms in the Navier-Stokes equation:

$\partial u/\partial x \sim U/l$

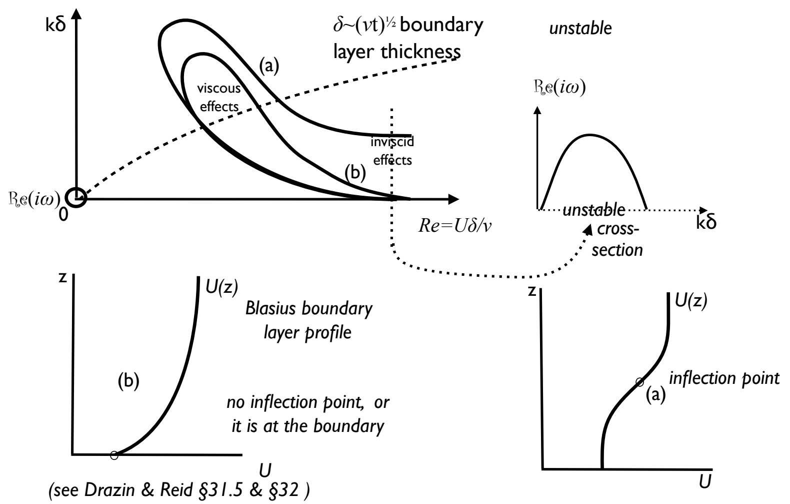
the inertia terms X:
$\rho\, u\, \partial u/\partial x \sim \rho\, U^2/l$

$\partial\tau/\partial y \sim \mu\, \partial^2 u/\partial y^2$ *(friction)*
*for laminar flow* = viscous terms
$\mu\, \partial^2 u/\partial y^2 \sim \mu\, U/\delta^2$

$$\mu\frac{U}{\delta^2} \sim \rho\frac{U^2}{l} \quad \Rightarrow \quad \delta \sim \sqrt{\frac{\nu l}{U}} \sim \sqrt{\nu t}$$

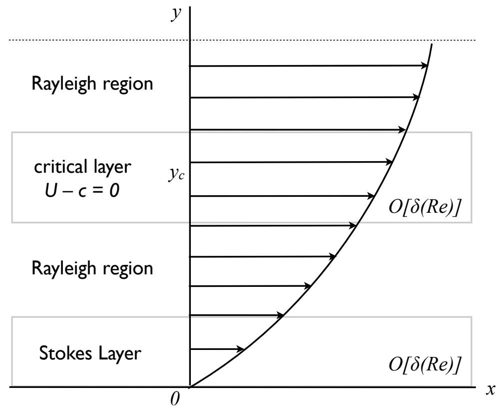
From the exact solution (see e.g. Batchelor 1969. or Schlichting p. 116
it can be shown that $\delta=5\sqrt{(\nu t)}$

# Parallel flow instability with Reynolds number



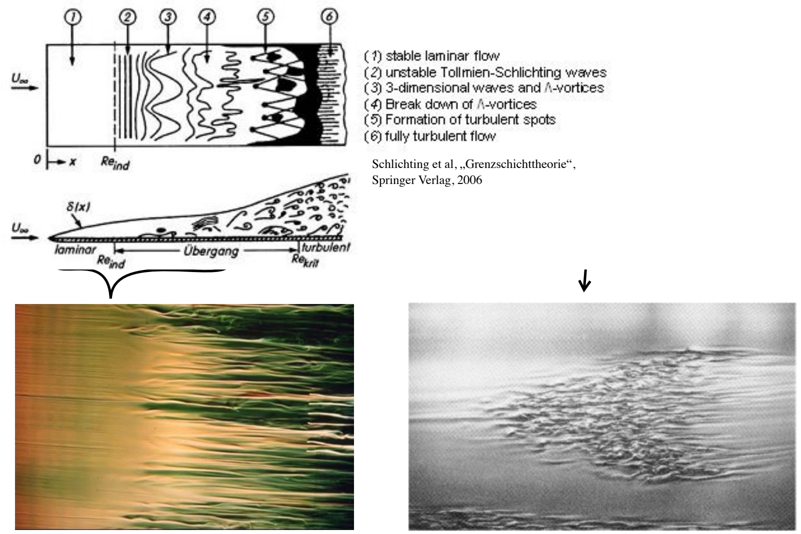

(see Drazin & Reid §31.5 & §32 )

Tolmien '29 $Re_c$=420; Schlichting '33 $Re_c$=575, Lin '45 421; Jordinson '70: $Re_c$=520




the Rayleigh regions are governed by the Rayleigh equation;
the Stokes and critical layer by viscous effects

# V*iscous effects* and the turbulent boundary layer

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Cantwell, et al (1978)

# Energy equation of the perturbed flow

Basic flow + $u_i$; multiply the NS equations with $u_i$'…. (bar for spatial average) …
which for two dimensional flow these equations reduce to (long calculations)
(primes are omitted)

$$\frac{1}{2}\frac{\partial \overline{u^2}}{\partial t} + \frac{1}{2}U\frac{\partial \overline{u^2}}{\partial x} + \frac{1}{2}V\frac{\partial \overline{u^2}}{\partial y} = -\overline{uv}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\left(\frac{1}{2}\overline{u^2 v} + \frac{1}{\rho}\overline{pv}\right) + \nu \overline{u_i \frac{\partial^2 u_i}{\partial x_j^2}}$$

T **A** C **B** D

A is the transfer of energy by the mean flow and B the transfer of energy by the perturbations.
When integrated over the entire flow domain, A and B become zero.

C input of energy via the shear stress (also called the Reynolds Stress)
D energy dissipation due to viscous effects
and thus we can write the equations as a balance between three terms T, C and D

Consider the instability for a region of just one wavelength long.
We can thus write

Kinetic energy

$$E = \frac{\rho}{2}\iint \overline{(u'^2 + v'^2)}dxdy$$

Reynolds shear stress
Work done against the mean shear flow

$$\rho M = -\rho\lambda \iint \overline{u'v'}\frac{dU}{dy}dxdy$$

Dissipation
(with vorticity $\zeta$' and $\mu=\nu/\rho$)

$$\mu N = \mu \iint \overline{(\zeta'^2)}dxdy$$

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}$$

the mechanical balance between the redistribution of energy by the Reynolds shear stress and the dissipation term N is:

$$\frac{\partial E}{\partial t} = \rho M - \mu N$$

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The sign of the right-hand side of this equation determines whether the energy of the disturbance increases or decreases. Large μ implies stable flow.

The ratio of the two terms can be written as

$$\rho M/_{\nu N} = Re\, \mathbf{M'}/_{\mathbf{N'}}$$

*Re* the Reynolds number and M' and N' the dimensionless form of M and N

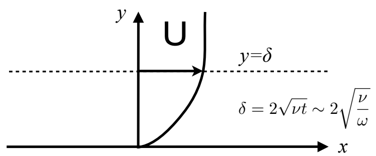
When *Re M'/N'* <1, i.e. when the Reynolds number is below the ratio M'/N', the disturbance will dissipate.

The least value of *N'/M'* yields a critical Reynolds number, $Re_c$ above which the flow will be unstable

illustration: instability of the Stokes boundary layer

# Stokes boundary flow

In some cases, the viscosity in the boundary layer is capable of rendering the flow unstable. We consider the example of C.C. Lin (1955).
Calculate the Reynolds stress for the vorticity equation of an oscillating boundary, of frequency ω.




$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\frac{\partial \bar{\zeta}}{\partial y} = \nu\left(\frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial y^2}\right)$$

Perturbations of the form $\sim e^{i(kx-\omega t)}$

Estimate the different advection terms inside the Stokes layer

## Stokes boundary flow

$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\frac{\partial \bar{\zeta}}{\partial y} = \nu\left(\frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial y^2}\right)$$

Perturbations of the form $\sim e^{i(kx-\omega t)}$

Estimate the different advection terms:
Inside the Stokes layer is $\delta = \delta_0$ so that $\zeta' \approx -\partial u'/\partial y$ we obtain $\zeta'\delta_0 \sim u'$.
with continuity du/dx+dv/dy=0 so that $v' \sim k\,\delta_0 u' = k\,\delta_0^2\,\zeta'$
Comparing the two advection terms to the time derivative:

$$\frac{U\zeta'_x}{\zeta'_t} \sim \frac{Uk\zeta'}{\omega\zeta'} \sim \frac{U}{C}$$

(writing derivatives as indices, and $C=\omega/k$, and $\zeta_y' \sim U/\delta^2$)

$$\frac{v'\bar{\zeta}_y}{\zeta'_t} \sim \frac{k\delta_0^2 U\zeta'}{\omega\zeta'\delta^2} \sim \frac{U}{C}\frac{\delta_0^2}{\delta^2}$$

In the boundary layer $U$ is small —> $U/C<<1$
both advection terms can therefore be neglected !
Since $\zeta'_{xx}/\zeta'_{yy} \sim (k\delta)^2<<1$ we consider only $y$ dependence:

$$\boxed{\frac{\partial \zeta'}{\partial t} = \nu\frac{\partial^2 \zeta'}{\partial y^2}}$$

## Stokes boundary flow

With the boundary conditions:

$y=0$ $\quad \zeta'=Ae^{i(kx-\omega t)}$ with $A$ the oscillation amplitude
$y \longrightarrow \infty$, $\quad \zeta' \longrightarrow 0$

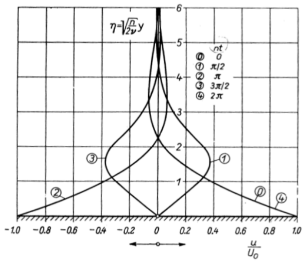
the solutions are then given by

$$\boxed{\zeta' = A\,e^{-y/\delta}\;e^{i(y/\delta+\theta)} = A\,e^{-y/\delta}\;e^{i\,(y/\delta + kx - \omega t)}}$$

oscillatory motion
damping with height

for $y=0$ we have a sinusoidal motion, for large $y$ the motion amplitude tends to zero

rewrite: $\zeta' = A\,e^{i\theta}\,e^{\alpha y}$ with $\boldsymbol{\alpha = (-1+i)/\delta_0}$ and $\boldsymbol{\theta = (kx - \omega t)}$




**$\delta$ is the depth of penetration of the viscous wave**

Schlichting p72-76

Fig. 5.8. Velocity distribution in the neighbourhood of an oscillating wall

The velocities can be obtained

$$u' = -\int_0^y \zeta' dy$$

and using continuity

$$v' = -\int_0^y u'_x dy$$

## Stokes boundary flow

$\boldsymbol{\alpha = (-1+i)/\delta_0}$ and $\boldsymbol{\theta = (kx - \omega t)}$

**shear stress ...**

$$u' = -\int_0^y \zeta' dy = -Ae^{i\theta}\int_0^y e^{\alpha y} = \frac{A}{\alpha}e^{i\theta}(1-e^{\alpha y})$$

$$v' = -\int_0^y u'_x dy = -\frac{ikA}{\alpha}e^{i\theta}\int_0^y (1-e^{\alpha y})dy = -\frac{ikA}{\alpha}e^{i\theta}\left[y - \frac{1}{\alpha}(e^{\alpha y}-1)\right]$$

In the boundary layer we can approximate, with $y/\delta<<1$ (and thus $\alpha y<<1$)
using Taylor expansions, we can write

$$u' \cong -\frac{A}{\alpha}e^{i\theta}\left(\alpha y + \frac{1}{2}\alpha^2 y^2 + ...\right) \qquad v' \cong \frac{ikA}{\alpha}e^{i\theta}\left(\frac{1}{2}\alpha^2 y^2 + \frac{1}{6}\alpha^2 y^3 + ...\right)$$

The Reynolds stress is defined as $\overline{u'v'}$. The time average over a period of these two sinusoidal functions can be written as … >>>

$$-\overline{u'v'} = -\frac{1}{2}\{ \qquad ... \qquad \} = \frac{k|A|^2}{24\delta}y^4 + ...h.o.t.$$

$$-\rho\overline{u'v'} = \frac{k}{\delta}\frac{|A|^2}{24}y^4 > 0 \quad \rightarrow \quad \boxed{M > 0}$$

Note that $u'$ and $v'$ have their origin from viscous effects. For sufficiently small viscous effects , the induced shear stress will be large enough ($\rho M > \mu N$) and thus provoke instability

$$\frac{\partial E}{\partial t} = \rho M - \mu N$$

How ?

## Stokes second problem)

see e.g.Schlichting p72-76

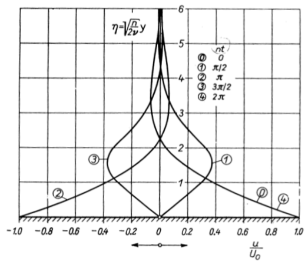



Fig. 5.8. Velocity distribution in the neighbourhood of an oscillating wall

In the oscillating frame of reference we can write the horizontal velocity in the form of (suppose $\theta=\omega t$):

oscillating plate
no slip at the wall:
for t>0 y=0 $u(0,t)=U_0 \sin nt$

consider solutions of the form: $u(y)= U_0\, exp[i(ky-nt)]$
this gives $\quad in=\nu k^2 \quad$ so that $\quad k=(i+1)/\delta$

solution: $u(y) = U_0\, e^{-y/\delta}\, e^{i(y/\delta - nt)}$

damping

for y=0 we recover the sinusoidal motion

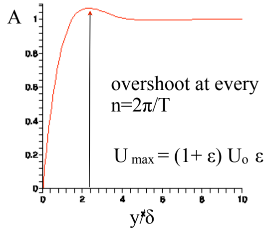
In the moving (oscillating) frame of reference:

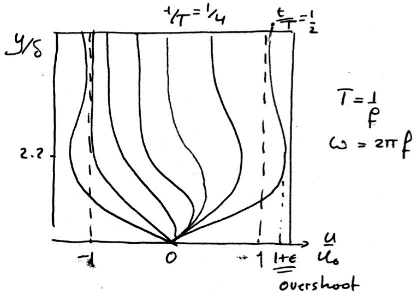
solution: $u(y) = U_0 \sin(nt) - U_0 \sin(nt - y/\delta)\, e^{-y/\delta}$

or: $u(y) = U_0 A \sin(nt + \psi)$ $\qquad A = [1 - 2e^{-y/\delta}\cos y/\delta + e^{-2y/\delta}]^{1/2}$

$$\psi = tan^{-1}\left[\frac{e^{-y/\delta}\sin y/\delta}{1 - e^{-y/\delta}\cos y/\delta}\right]$$

$A = [1 - 2e^{-y/\delta}\cos y/\delta + e^{-2y/\delta}]^{1/2}$




after one oscillation the maximum speed will exceed the amplitude $U_0$

$$\text{with } \bar{u} = \frac{u}{U_0} \text{ and } \tau = nt$$

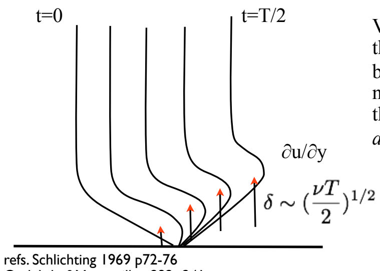
$$\frac{\partial \bar{u}}{\partial \tau} = \frac{\partial}{\partial \tau}[\sin\tau - \sin(\tau - y/\delta)e^{-y/\delta}] = \cos\tau - \cos(\tau - y/\delta)e^{-y/\delta}$$

consider the vorticity ∂u/∂y of this flow

$$\zeta = \frac{\partial \bar{u}}{\partial y} = \frac{1}{\delta}\left[\cos(\tau - y/\delta) + \sin(\tau - y/\delta)\right] e^{-y/\delta}$$

after half a cycle τ=T/2 and a diffusion time $\frac{\delta^2}{\nu} \sim \frac{T}{2}$ at the corresponding height

the velocity perturbation due the vorticity is in phase with the velocity variation




refs. Schlichting 1969 p72-76
Godrèche&Manneville p232- 241

Viscosity gives the fluids a memory:
the inertia given to the fluid an instant before has not disappeared; a cycle later, newly generated motion is added;
this changes the phase $\varphi$ at the level $y$
*accumulative effect in time*

another way to see this is to consider the pressure:

far from the plate we have $\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ so that $\frac{\partial p}{\partial x} \sim inu$

$$\frac{\partial}{\partial t}\bar{u} = -\frac{\partial p}{\partial x} + \frac{\partial^2 \bar{u}}{\partial (y/\delta)^2}$$

the pressure variation is transmitted *instantaneously*
after Δt=T/2 at a height $\delta^2 \sim \nu T/2$ the two ($p$ and $u$) will be in phase

# Orr-Sommerfeld equation

The above example is a heuristic demonstration, and not a full analytical proof, for which one should consider the Orr-Sommerfeld equation:

$$(U - c)(\phi'' - k^2\phi) - U''\phi = \frac{1}{ik\,Re}\left[\phi'''' - 2k^2\phi'' + k^4\phi\right]$$

+ appropriate boundary conditions for $\phi$

which include viscous effects, and so there is dissipation due to viscous effects. Also the critical layer appear for the same values $y_c$ but is now opposed by viscous effects.

Approaches include often asymptotic analyses and eigenvalues for instability are often calculated numerically and has been considered by Heisenberg 1924, Schlichting 1933 and Tollmien 1947. This is reported in the book of Lin 1955, and Drazin & Reid 1981chapters 4 and 5, and is left for further reading.