ROTATING FLUIDS

ROTATING FLUIDS

- Geophysical flows :
  Earth Oceans and atmosphere, mantel convection,

- Astrophysical flows,
  Other planetary atmospheres, accretion disks

- Industrial flows :
  mixing of chemical compounds in rotating containers,
  centrifuges in nuclear power industry
  coating of material on disks (so called Spin-coating)
  etc.

ROTATING FLUIDS

subjects
* Taylor Proudman theory

* The Ekman layer; Ekman layer instability

* Inertial waves (Rossby waves)

* Shallow water equations
  Barotropic instability

  Rayleigh criterion circular flow (vortices)

  Baroclinic instability

ROTATING FLUIDS (intro)

We consider a homogenous fluid in *solid body rotation* with density ρ and viscosity ν, and use the Cartesian coordinate system

$$\bar{\mathbf{x}} = x\bar{\mathbf{i}} + y\bar{\mathbf{j}} + z\bar{\mathbf{k}}$$

with the rotation vector in the $\mathbf{k}$ direction, the rotation vector is $\mathbf{\Omega} = \Omega_z\,\mathbf{k} = (0, 0, \Omega_z)$. Suppose a point of mass m at a position $\mathbf{x}$ experiences a force $\mathbf{F}$, then according to the second law of Newton :

$$m\left[\frac{d^2\bar{\mathbf{x}}}{dt^2} + 2\bar{\Omega}\times\frac{d\bar{\mathbf{x}}}{dt} + \bar{\Omega}\times\bar{\Omega}\times\bar{\mathbf{x}}\right] = \bar{\mathbf{F}}$$

resp. acceleration, Coriolis force and centrifugal force

Suppose $\mathbf{u}(\mathbf{x},t)$ is the fluid velocity with respect to the inertial system, the Navier-Stokes equations is with

$$\bar{\Omega}\times(\bar{\Omega}\times\bar{\mathbf{x}}) = -\nabla(\tfrac{1}{2}\Omega^2 r^2),\quad r = \sqrt{x^2+y^2}$$

$$\rho\left[\frac{\partial\bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} + 2\Omega\times\bar{\mathbf{u}} - \nabla(\frac{1}{2}\Omega^2 r^2)\right] = -\nabla p + \rho\bar{\mathbf{g}} + \mu\nabla^2\bar{\mathbf{u}}$$

ROTATING FLUIDS

For ρ =constant, the gravity force can be written as a potential V, so that the pressure ***p*** can be defined as

$$p = \frac{p}{\rho} + V - \frac{1}{2}\Omega^2 r^2$$

we obtain for the Navier Stokes equation in a rotating fluid:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + 2\Omega \times \bar{\mathbf{u}} = -\nabla p + \nu \nabla^2 \bar{\mathbf{u}}$$

and continuity (since ρ=constant) : $\nabla \cdot \bar{\mathbf{u}} = 0.$

Note : The third lhs term is the Coriolis force.
There is no distance ***r*** with respect to the axis of rotation,
The position of the axes ***r=0*** has no importance for the Coriolis force.

ROTATING FLUIDS

In order to make the NS equation dimensionless we often scale with a characteristic velocity $\mathbf{U_0}$, length $\mathbf{L}$, time scale $\mathbf{L/U}$ and pressure scale $\mathbf{LU^2}$.

$$\bar{\mathbf{x}} = L\bar{\mathbf{x}}', \; t = Tt', \; \bar{\mathbf{u}} = U\bar{\mathbf{u}}', \; \bar{\mathbf{p}} = \rho\Omega L U p', \; \bar{\Omega} = \Omega_z \bar{\mathbf{k}}$$

After substitution (after omitting the primes) :

$$\frac{1}{\Omega T}\frac{\partial \bar{\mathbf{u}}}{\partial t} + Ro(\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + 2\bar{\mathbf{k}} \times \bar{\mathbf{u}} = -\nabla p + E\nabla^2 \bar{\mathbf{u}}$$

$$\nabla \cdot v = 0$$

The non dimensional numbers Ro, E and ΩT are :

$$Ro = \frac{U}{\Omega L} = \frac{U^2/L}{\Omega U} \sim \frac{\textit{inertia forces}}{\textit{Coriolis force}} \quad (\textit{Rossby number})$$

$$E = \frac{\nu}{L^2\Omega} = \frac{\nu U/L^2}{\Omega U} \sim \frac{\textit{viscous effects}}{\textit{Coriolis force}} \quad (\textit{Ekman number})$$

T, is the characteristic flow time $\quad \Omega T = \frac{2\pi T}{2\pi/\Omega}$

ROTATING FLUIDS
Geophysical large scale flows

In order to simplify the NS equation, we can consider the order of magnitude of these numbers for a particular system.

For example consider a large scale geophysical flow the scales are very large O(4000Km), velocities of O(20m/s) (JetStream) and the background rotation f ≈ 10−4. For the Ro-number this implies

$$Ro \approx \frac{20}{4\,10^6\,10^{-4}} = \frac{1}{20} \approx 0.05, \text{ i.e. } Ro << 1.$$

$$(\Omega T)^{-1} << 1.$$

The Ekman number is small in fast rotating and large scale flows
(i.e. viscous effects are small, except in the boundary)

$$E << 1.$$

The leading order (dimensionless) geostrophic balance equations are:

$$2\bar{\mathbf{k}} \times \bar{\mathbf{u}} = -\nabla p \qquad \nabla \cdot \bar{\mathbf{u}} = 0$$

i.e. balance between the pressure and the coriolis force. (P is constant along stream lines).

ROTATING FLUIDS
Taylor Proudman theorem

Eliminate the pressure term by taking the curl of $2\bar{\mathbf{k}} \times \bar{\mathbf{u}} = -\nabla p$

$$2\nabla \times (\bar{\mathbf{k}} \times \bar{\mathbf{u}}) = -\nabla \times (\nabla p) = 0$$

with the vector identity

$$\nabla \times (\bar{\mathbf{k}} \times \bar{\mathbf{u}}) = \bar{\mathbf{k}}(\nabla \cdot \bar{\mathbf{u}}) + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{k}} - \bar{\mathbf{u}}(\nabla \cdot \bar{\mathbf{k}}) - (\bar{\mathbf{k}} \cdot \nabla)\bar{\mathbf{u}}$$

and continuity $\qquad \nabla \cdot \bar{\mathbf{u}} = 0$

$$\bar{\mathbf{k}} \cdot \nabla \bar{\mathbf{u}} = 0 \qquad \rightarrow \qquad \frac{\partial \bar{\mathbf{u}}}{\partial z} = 0$$

This is the Taylor-Proudman theorem stating that, to leading order (Ro«1), there are no variations in the velocity along the axes of rotation

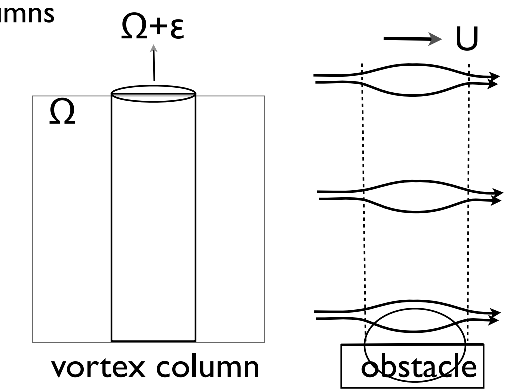
(Proudman 1916, Taylor 1923).

ROTATING FLUIDS
Taylor Proudman theorem

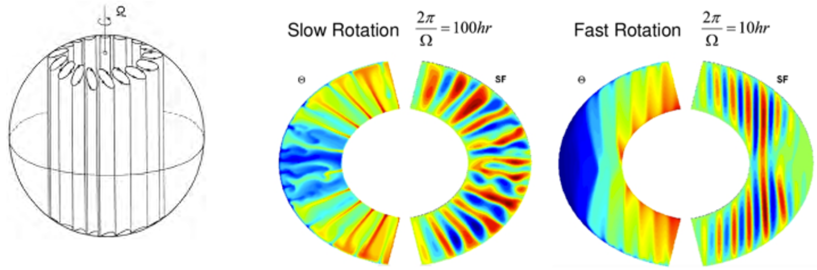
When the flow is confined between boundaries perpendicular to the rotation axes, then, since u = v = w = 0 at these boundaries

$\partial \mathbf{u}/\partial z = 0$ $\quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0,$ to leading order 2D flow!

=> Taylor columns

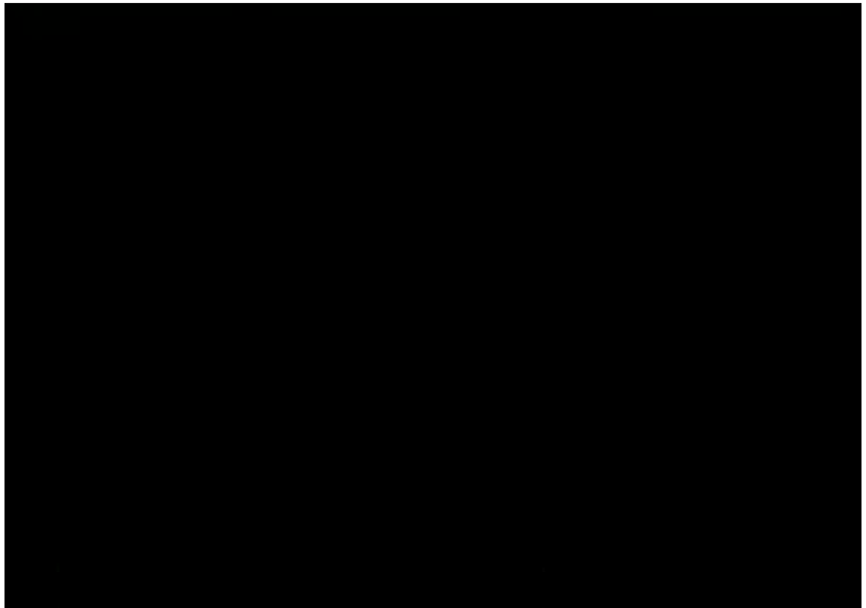


ROTATING FLUIDS
Taylor columns

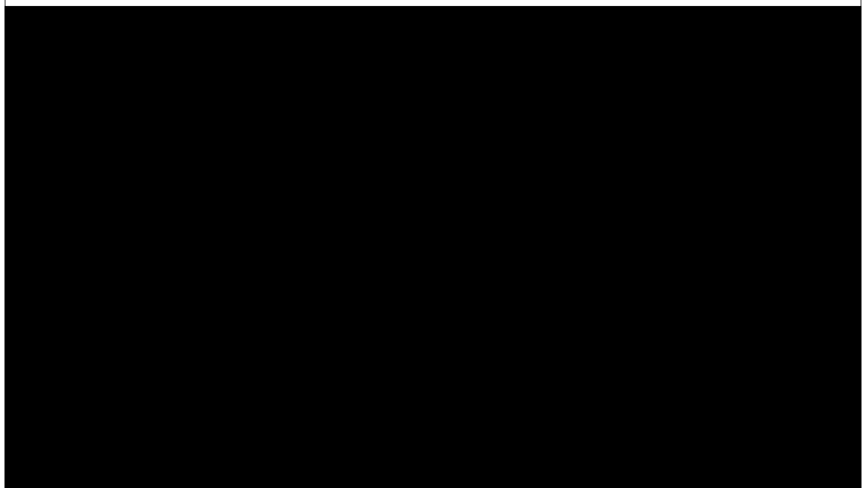


Busse 1970

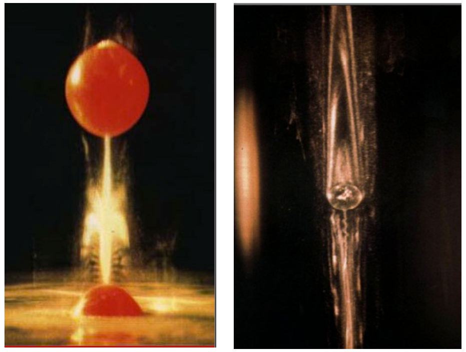
Internal convection on a gas giant using MITjcm – High Taylor number simulations are dominated by Taylor columns parallel to the axis of rotation (image source – Yohai Kaspi)



https://www.youtube.com/watch?v=Ans3tnvMyTk&list=PL0EC6527BE871ABA3&index=19&feature=plpp_video



ROTATING FLUIDS
Taylor columns



rising sphere for Ro<<1

ROTATING FLUIDS
inertial waves

In the geostrophic approximation Ro<<1
the equations reduce to the linear relation:

$$\nabla.\bar{\mathbf{u}} = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + 2\mathbf{\Omega} \times \bar{\mathbf{u}} = -\frac{1}{\rho}\nabla p$$

$$\longrightarrow \quad \frac{\partial^2 \nabla^2 w}{\partial t^2} + 4\Omega^2 \frac{\partial^2 w}{\partial z^2} = 0$$

This equation has planar waves as solutions:

$$\bar{\mathbf{u}} = Re(\mathbf{A})e^{i(\bar{\mathbf{k}}\cdot\bar{\mathbf{x}} - \sigma t)}$$

$$p = Re(P)e^{i(\bar{\mathbf{k}}\cdot\bar{\mathbf{x}} - \sigma t)}$$

$\mathbf{k}$ the wave vector
and σ the frequency

Substitution gives

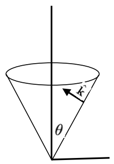
$$\left(\sigma^2\mathbf{k}^2 - 4\Omega^2 m^2\right) w = 0$$

dispersion relation of inertial waves

$$\sigma^2 = \frac{4\Omega^2 m^2}{\mathbf{k}^2} = \frac{4\Omega^2 m^2}{k^2 + l^2 + m^2}$$

There is a relation between frequency σ, and θ
the angle of the propagation with the rotation axis

$$\frac{\sigma}{2\Omega} = \cos\theta$$

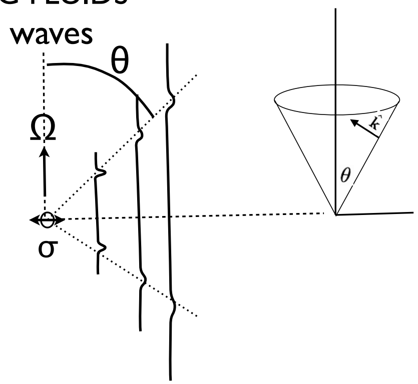


ROTATING FLUIDS
inertial waves

$$\frac{\sigma}{2\Omega} = \cos\theta$$

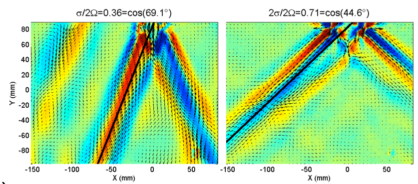
$$c_p = \frac{\sigma}{|\bar{\mathbf{k}}|}\hat{\mathbf{k}}$$

$$c_g = \frac{2}{|\bar{\mathbf{k}}|}\bar{\mathbf{\Omega}} - \bar{c}_p$$



≈ analogy with Internal waves
in stratified fluids
(restoring force is gravity)

Inertial waves are
also called
gyroscopic waves
(restoring force=Coriolis force)



Inertial oscillation

$$\frac{V^2}{R} + fV = -\frac{1}{\rho}\frac{dp}{dn}$$

When $dp/dn = 0$ there is a balance between Coriolis force and centrifugal force.

Fluid parcels move along a circular path with radius $\mathbf{R = -V/f}$
i.e. parcels move in *anticyclonic* direction

The motion is described: $u(t) = V\cos ft\ , \quad v(t) = -V\sin ft$

and $V = (u^2 + v^2)^{1/2}$

The oscillation has a period $T = 2\pi/f$

See experiment later on.

EKMAN BOUNDARIES

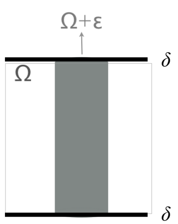
Ekman boundary layers

Ekman Boundary layers

Consider a geostrophic flow. In the interior the Taylor Proudman theorem holds, so that $\bar{\mathbf{U}}_\mathbf{I} = U_I(x,y)$ and $P_I = P_I(x,y)$. At the boundaries there is adjustment to zero velocity ($\bar{\mathbf{u}} = 0$) so that $\frac{\partial \bar{\mathbf{u}}}{\partial z} \neq 0$, and nonzero vertical velocities.

This thin layer is called the Ekman layer :

Vertical gradients are large : $\frac{\partial}{\partial z} >> 1$, or $\frac{\partial}{\partial z} >> 1/\delta$ and $\delta << 1$.
In the boundary, we have (E for Ekman boundary layer)



$$\begin{aligned} -2v_E &= -\frac{\partial p_E}{\partial x} + E\frac{\partial^2 u_E}{\partial z^2} \\ 2u_E &= -\frac{\partial p_E}{\partial y} + E\frac{\partial^2 v_E}{\partial z^2} \\ 0 &= \frac{\partial p_E}{\partial z} + E\frac{\partial^2 w_E}{\partial z^2} \\ \frac{\partial u_E}{\partial x} &+ \frac{\partial v_E}{\partial y} + \frac{\partial w_E}{\partial z} = 0 \end{aligned}$$

Ekman boundary layers

Ekman Boundary layers

Since $w_E = O(\delta) << 1$, $w(E)/\delta = O(1) \Longrightarrow \frac{P_E}{\delta} + E\frac{w_E}{\delta^2} = 0$ (E<<1)
i.e.

$$\frac{\partial p_E}{\partial z} = 0$$

Thus the pressure in the Ekman layer P_E must be equal to the pressure in the interior, P_I for which we know that

$$-2v_I = -\frac{\partial P_I}{\partial x} \quad 2u_I = -\frac{\partial P_i}{\partial y}$$

so that

$$-2v_E = -2v_I + E\frac{\partial^2 u_E}{\partial z^2} \quad 2u_E = 2u_I + E\frac{\partial^2 v_E}{\partial z^2}$$

We solve this by defining a complex velocity
$\phi = (u_E + iv_E) - (u_I + iv_I)$ so that

$$E\frac{\partial^2 \phi}{\partial z^2} = 2i\phi$$

Ekman boundary layers

Ekman Boundary layers

With the boundary conditions

$$z = 0: \quad \phi = -(u_I + iv_I)$$
$$z/\delta \to \infty: \quad \phi \to 0$$

the solution is $\phi = -(u_I + iv_I)\exp(-E^{-\frac{1}{2}}(1+i)z)$ with

$$u_E = u_I + \exp(-E^{-\frac{1}{2}}z)[-u_I\cos(E^{-\frac{1}{2}}z) - v_I\sin(E^{-\frac{1}{2}}z)]$$
$$v_E = v_I + \exp(-E^{-\frac{1}{2}}z)[u_I\sin(E^{-\frac{1}{2}}z) - v_I\cos(E^{-\frac{1}{2}}z)]$$

Ekman layer thickness is $\delta = E^{\frac{1}{2}}L = \sqrt{\nu/\Omega}$.

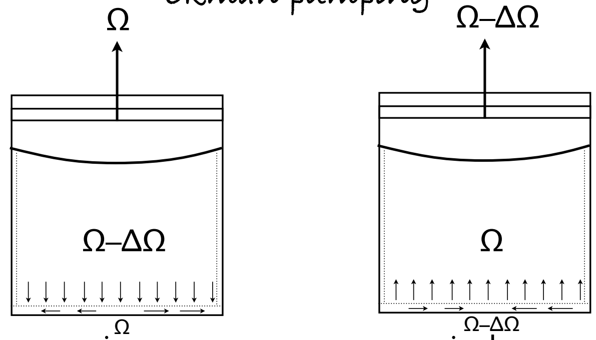
Ekman pumping (with continuity $\nabla \cdot U_E = 0$ and $\nabla_h \cdot U_I = 0$)

$$w_I = \tfrac{1}{2}E^{\frac{1}{2}}\omega_I$$

with ω_I the vorticity in the interior, and when the bottom rotates :

$$w_I = \frac{1}{2}E^{\frac{1}{2}}(\omega_I - \omega_b)$$

Ekman pumping



spin up

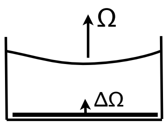
bottom rotates ΔΩ ***faster*** than the fluid and spins it up by accelerating the fluid in the Ekman boundary layer, vortex stretching Ekman pumping to the boundary causes vortex stretching.

spin down

bottom rotates ΔΩ ***slower*** than the fluid and spins it down. Ekman pumping into the interior cause vortex squeezing, leading to slower rotation

Einstein tea leaves ....

Ekman boundary layers



Ro=ΔΩ/Ω <<1

For convenience, consider a rotating body of fluid in which a disk rotates relatively to the rotating frame of reference. We obtain for the non-dimensionalized (z/δ, and $\frac{U}{V_\infty}$) analytic solution :

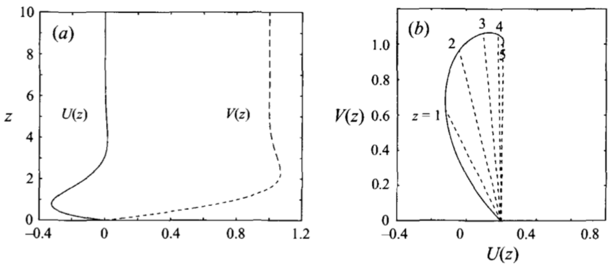
$$U(z) = \frac{U}{V_\infty} = -e^{-z}\cos z$$

$$V(z) = \frac{V}{V_\infty} = 1 - e^{-z}\cos z$$

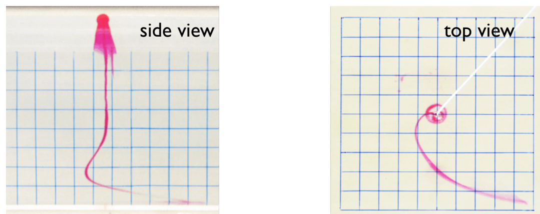
Ekman spiral for a faster rotating bottom

Mean velocity profiles for the Ekman layer flow.

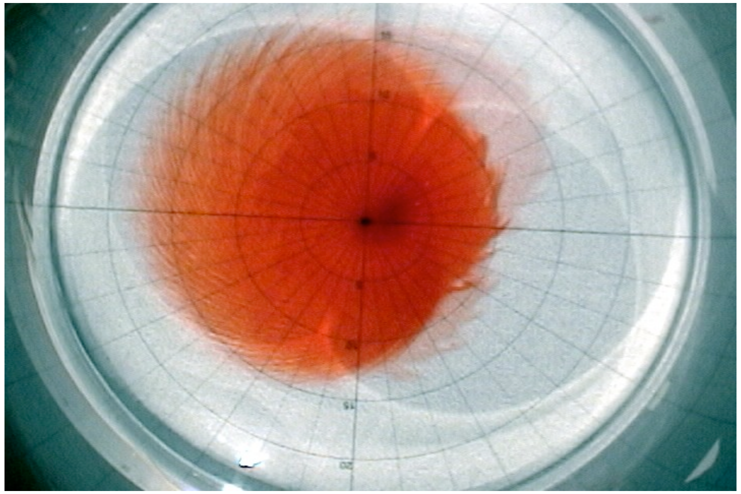
Mean velocity profiles plotted as an Ekman spiral.



dye drop falling verically in the Ekman layer



for a vortex (spirals at the bottom)



Greenspan. 1961

Ekman Layer instability

fig. 6.6. The local orientation of these waves is anywhere from 0° to −8° with respect to the geostrophic flow; the wave lengths vary between 25δ and 33δ and the phase speed is approximately $0{\cdot}16V_I$, directed radially inward. Fig. 6.7 shows the critical Reynolds number plotted versus the Rossby number; the relationship is approximately

$$R_E^{(A)} = 56{\cdot}3 + 58{\cdot}4\epsilon^{(A)}. \quad (6.3.3)$$

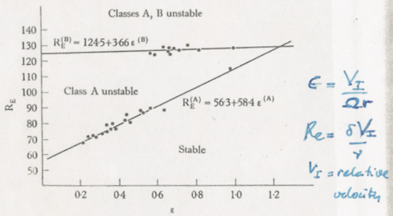


Fig. 6.7. The critical Reynolds number vs. Rossby number for Class A and Class B instabilities[199].

Waves of this family develop first and are very sensitive to the value of ε. As ε increases, the disturbance ceases to be confined to the boundary layer and the effects propagate throughout the interior, fig. 6.8, much like the process pictured in fig. 6.4 of the last section. Since the primary frequency of the disturbance is greater than 2Ω, a non-linear wave interaction within the boundary layer resulting in a lower frequency wave may be responsible for the interior excitation.

Class B waves form an angle of almost exactly 14·6° with respect to the geostrophic flow. The wavelength is $11{\cdot}8\delta$ and the phase velocity is $0{\cdot}034V_I$ directed radially inward. There is only a slight dependence of critical Reynolds number on the Rossby number given b

$$R^{(B)} = 124{\cdot}5 + 3{\cdot}66\epsilon^{(B)} \quad (6.3.4)$$

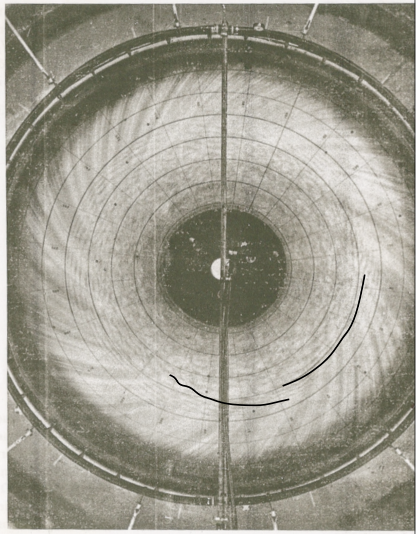


Fig. 6.6. A photograph by Faller and Kaylor [47] showing Class A waves and turbulence at smaller radii.

Ekman Layer instability

Indeed, the last cited reference was really the first thorough experimental and theoretical examination of Class B waves.

It is relatively easy to demonstrate that the transient Ekman layer can go unstable in either spin-up or spin-down. Permanganate crystals dropped about the periphery of a uniformly rotating cylindrical container produce a thin annular layer of dyed fluid at the bottom plate, fig. 6.15(*a*). During spin-down, this coloured

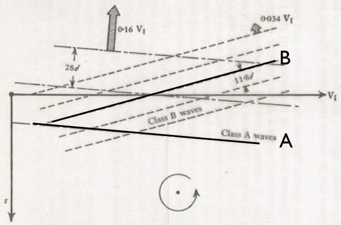
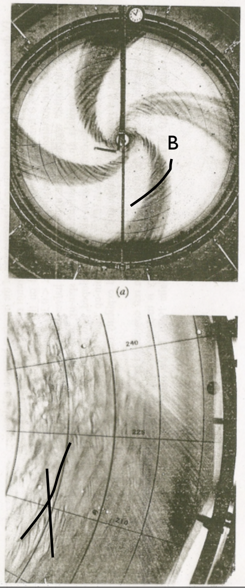


Fig. 6.10. Diagram of the two instability classes to illustrate the measured wave lengths, phase velocities and wave front orientations.

fluid is drawn radially inward by the efflux from the Ekman layer and within a few revolutions, a series of rolls develop, plate (*b*). (The classification of these waves, A or B, is uncertain.) The waves amplify considerably and by the time spin-down is achieved, the dyed fluid occupies an almost perfectly circular lens, of modest thickness, at the bottom of the tank. This 'disk' is separated from the outer wall by a ring of clear fluid drawn down from the vertical surface of the cylinder. (The remaining plates of fig. 6.15 concern a spin-down experiment with a stratified fluid to be discussed shortly.) Spin-up can exhibit the same type of instability but the techniques of visualization must be altered slightly.



(*a*)

Ekman Layer instability

The Ekman layer instability has been investigated experimentally and theoretically by Faller JFM 1963-1991 and theoretically by Lilly 1966 see Lingwood 1996, 1997

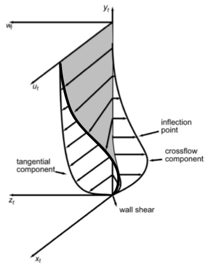


Figure 1 Swept-wing boundary-layer profiles.

The instability is related to inflection point instability Cross flow instability related cross flow over aircraft wings,

NOTE: Similar approaches for von Karman rotating disk flow, and Bödewadt flow (i.e. rotating fluid above a stationary disk) see Lingwood JFM 1996, 1997, Saric 2003 Annu. Ann. Rev. Fluid Mech. 2003. 35:413–40

MODELING OF ROTATING FLOWS

barotropic instability

Shallow water approximation (Geophysical large scale flow)

(suppose for simplicity $\rho =$ constant and $\nu = 0$)

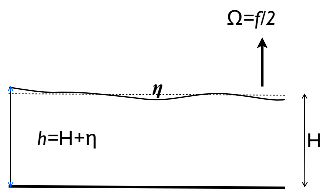
- shallow fluid with $L \ll H$, $w \ll u$
- hydrostatic balance
- Boussinesq approximation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$0 = -\frac{\partial p}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



shallow water layer with free surface

rewriting the Shallow water equations

Integration of the continuity equation from z=0 to $H + \eta$ gives :

$$(H+\eta)\frac{\partial u}{\partial x} + (H+\eta)\frac{\partial v}{\partial y} + w(\eta) + w(0) = 0$$

Since $w(0) = 0$ and $w(\eta) = \frac{D\eta}{Dt}$ we obtain :

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[u(H+\eta)] + \frac{\partial}{\partial y}[v(H+\eta)] = 0$$

with $h = H + \eta$ this can be written as

$$\frac{Dh}{Dt} + h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

.

Potential vorticity equation — Intermezzo

Cross differentiation and substitution of the continuity in the Euler relations directly gives for the vorticity $\bar{\omega} = (0, 0, \omega_z)$, with $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$:

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} + f(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

with

$$\frac{Dh}{Dt} + h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0.$$

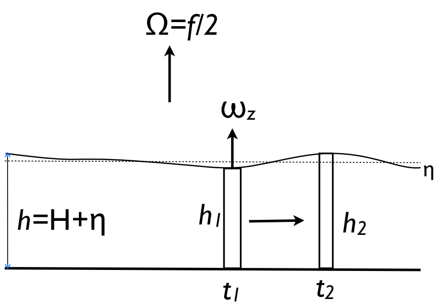
This gives the conservation of potential vorticity :

$$\frac{D\omega_z}{Dt} = \frac{\omega_z + f}{h}\frac{Dh}{Dt}$$

or

$$\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0$$

In a shallow layer



$$\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0$$

increase in h involves a change in relative vorticity ω_z (increase) in order to keep the PV constant (note that f = $f_0 + \beta y + \gamma y^2$ ...)

Shallow water equations and barotropic Instability !

Since $L \ll H$ and $w \ll u$, $\frac{\partial w}{\partial z} = 0$
Since $\nu = 0$, u and v don't vary in z direction : $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$.

As a basic state we consider a uniform flow : $u = U(y)$ that satisfies geostrophic balance, i.e. : $fU = -\frac{1}{\rho_0}\frac{dP}{dy}$
Perturbations :

$$\begin{aligned} u &= U(y) + u'(x, y, t) \\ v &= v'(x, y, t) \\ p &= P(y) + p'(x, y, t) \end{aligned}$$

$$\begin{aligned} \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} - fv' &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\ \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + v'\frac{\partial U}{\partial y} + fu' &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 \end{aligned} \quad (2)$$

The last equation admits the use of a streamfunction

$$u' = -\frac{\partial \psi}{\partial y} \ , \ v' = \frac{\partial \psi}{\partial x}$$

so that after cross differentiation we obtain :

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \frac{d}{dy}\left(f - \frac{dU}{dy}\right)\frac{\partial \psi}{\partial x} = 0$$

($\nabla^2\psi = \omega$, and so deriving the vorticity yields the same result)

A perturbation of the form $\psi = \phi(y)\exp[i(lx - \omega t)]$ yields :

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\frac{d}{dy}(f - \frac{dU}{dy})}{U - c}\phi = 0 \quad c = \omega/l$$

in which we recognize the Rayleigh equation. It is too difficult to solve this system for its unstable eigenvalues, and we consider the less constraining, integral properties. As for Rayleigh's criterion multiply the ϕ with its complex conjugate ϕ^* to get with boundary conditions $\phi(y=0) = \phi(y=L) = 0$:

$$-\int_0^L \left(|\frac{d\phi}{dy}|^2 + l^2|\phi|^2\right) dy + \int_0^L \frac{\frac{d}{dy}(f - \frac{dU}{dy})}{U - c}|\phi|^2 dy = 0$$

The imaginary part of this expression is :

$$c_i \int_0^L \frac{d}{dy}\left(f - \frac{dU}{dy}\right)\frac{|\phi|^2}{|U - c|^2} dy = 0$$

Stable flow when $c_i = 0$.

For $c_i \neq 0$ $\boxed{\frac{d}{dy}(f - \frac{dU}{dy})}$ must change sign for instability!

Note that for $f = 0$ we recover Rayleigh instability criterion for a shear layer with vorticity $\omega_z = \frac{d^2U}{dy^2}$.

Further we note that in reality f varies with latitude y., i.e.

$f = f_0 + \beta y$.

A background vorticity gradient thus changes the stability criterion.

Centrifugal instability

Rayleigh criterion for centrifugal instability was

$$\frac{d}{dr}(rv)^2 \geq 0$$

In a rotating fluid this criterion is

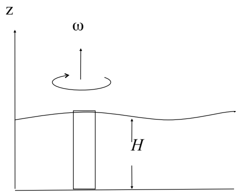
$$\frac{d}{dr}(rv + \frac{1}{2}fr^2)^2 \geq 0$$

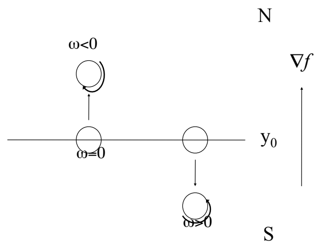
Or written with the vorticity

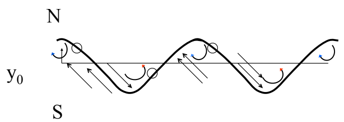
$$(v + \Omega r)(\omega + 2\Omega) \geq 0$$

intermezzo Rossby waves

$$\boxed{\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0} \qquad f = f_0 + \beta y$$







intermezzo Rossby waves example

$$\boxed{\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0} \qquad U=0 \quad \Rightarrow$$

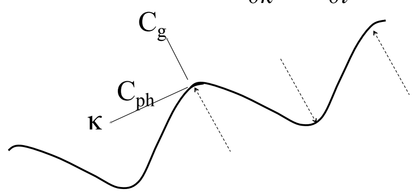
perturbation equation:
$\partial/\partial t\, \Delta\psi + \beta\, \partial\psi/\partial x = 0$

$\psi \sim A \exp[i(kx+ly - \omega t)]$

Dispersion relation $\quad \omega \cong -\frac{k\beta}{k^2 + l^2} \qquad \|K\| = \sqrt{k^2 + l^2}$

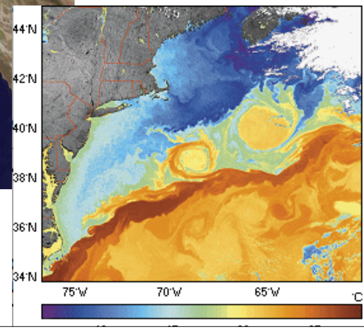
Phase speed: $\quad C_{ph} = \omega / k$

Group speed: $\quad \vec{C}_g = \frac{\partial \omega}{\partial k}\hat{i} + \frac{\partial \omega}{\partial l}\hat{j}$



Baroclinic Instability

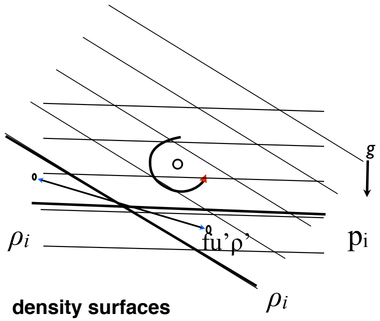
Baroclinic instability



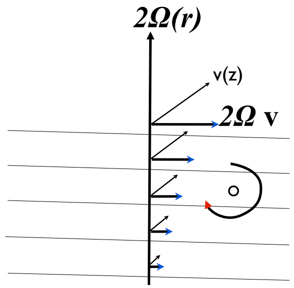
Baroclinic instability

basics

stratified fluid with vertical shear and rotation



density surfaces



Shear + Coriolis

thermal wind relation $-\frac{\nabla\rho \times \nabla p}{\rho^2} = 2\bar{\Omega}.\nabla\bar{u}$

$$\frac{1}{\rho^2}\left(\frac{\partial p}{\partial z}\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial z}\frac{\partial p}{\partial x}\right) = 2\Omega\frac{\partial v}{\partial z}$$

side view
Vertical plane

Slanting convection

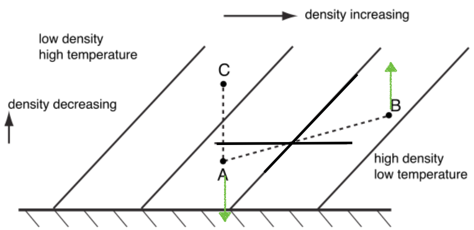


Fig. 6.9 A steady basic state giving rise to baroclinic instability. Potential density decreasing upwards and equatorwards, and the associated horizontal pressure gradient is balanced by the Coriolis force. Parcel 'A' is heavier than 'C', and so statically stable, but it is lighter than 'B'. Hence, if 'A' and 'B' are interchanged there is a release of potential energy.

a decrease in Potential Energy gives unstable motion

restoring force due to mass density excess

$$F_p = g\frac{\Delta \rho_{ab}}{\bar{\rho}} \sin \phi$$

within the wedge (red) :

$F_p<0$

i.e. fluid parcels are accelerated in the direction of the displacement. (Unstable), centre of mass is lower => potential energy is released.

Outside the wedge

$F_p>0$

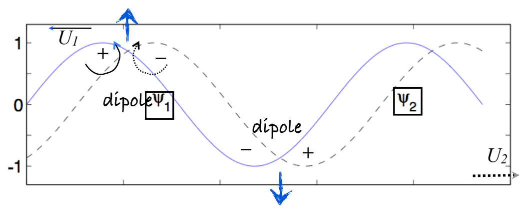
Particles move back to original position

Baroclinic instability

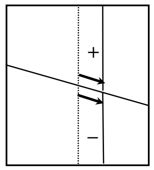
basics

top view: horizontal plane

side



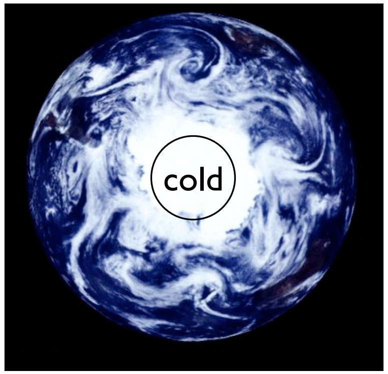
PV conservation



MODELS :

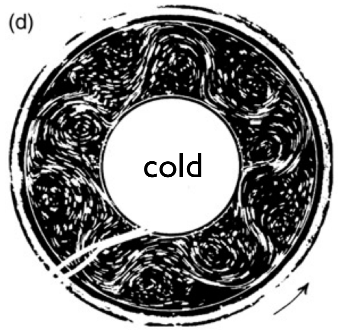
Phillips model, Eady model, Charney model,

Baroclinic instability Experiments



South Polar Projection of Earth
http://photojournal.jpl.nasa.gov/

(d)



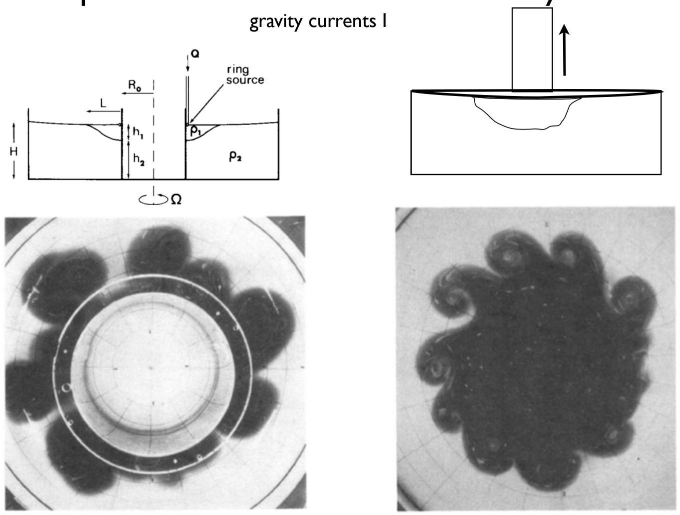
Ω- 3.64 rad s⁻¹

Regular baroclinic waves, m= 5

Hide GAFD 2005

Experiments on Baroclinic instability I

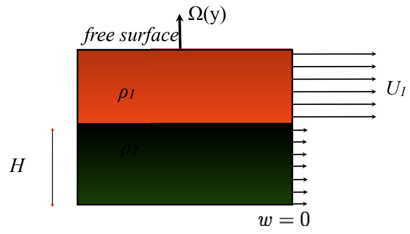
gravity currents I



$\sim R_d$ Deformation radius R_d=NH/f $\sim R_d$

Phillips (1954)

two-layer model



non-dimensional numbers:

Rossby number
Ro=U/fL or ω/2Ω <<1

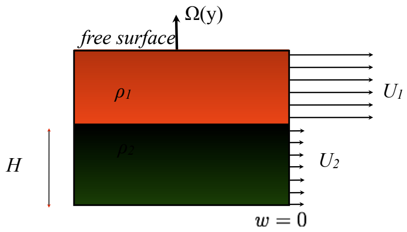
Froude number
$F = (R/R_d)^2$;

Deformation radius
R_d=NH/f

$$NH = \sqrt{\frac{g'}{H}}H = \sqrt{g'H}$$

Phillips (1954)

two-layer model



$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_1}\frac{\partial p}{\partial x} \quad \frac{Du}{Dt} - fv = -\frac{1}{\rho_2}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_1}\frac{\partial p}{\partial y} \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_2}\frac{\partial p}{\partial y}$$

$$f = f_0 + \beta y \qquad 0 = -\frac{\partial p}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$F = \frac{(2\Omega)^2 R^2}{g'H} = (R/R_d)^2$$

$$\frac{D}{Dt}[\omega_1 - F(\psi_1 - \psi_2) + \beta y] = 0$$

$$\frac{D}{Dt}[\omega_2 - F(\psi_2 - \psi_1) + \beta y] = 0 \qquad \Rightarrow$$

The coupling between the two layers is via the second term, and depends on the layer depth (pressure)

general procedure: perturbation equations ...

$$\Rightarrow \begin{cases} \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_1 + \frac{1}{2}F^2(\psi_2 - \psi_1)\right] + \frac{\partial \psi_1}{\partial x}(\beta + F^2U) = 0 \\ \left[\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_2 + \frac{1}{2}F^2(\psi_1 - \psi_2)\right] + \frac{\partial \psi_2}{\partial x}(\beta - F^2U) = 0 \end{cases}$$

$U_{1,2} = \overline{U} \pm U$

substitute $\psi_i = Re\Psi_i e^{i(kx+ly-\omega t)}$

$$\Rightarrow \begin{cases} [ik(U-c)]\left[-K^2\Psi_1 + \frac{1}{2}F^2(\Psi_2 - \Psi_1)\right] + ik\Psi_1(\beta + F^2U) = 0 \\ [-ik(U+c)]\left[-K^2\Psi_2 + \frac{1}{2}F^2(\Psi_1 - \Psi_2)\right] + ik\Psi_2(\beta - F^2U) = 0 \end{cases}$$

after substracting and adding the equations we get :

$$\Rightarrow \begin{cases} \left[(U-c)(\frac{1}{2}F^2 + K^2) - (\beta + F^2U)\right]\Psi_1 - [F^2(U-c)/2]\,\Psi_2 = 0 \\ [F^2(U+c)]\,\Psi_1 - \left[(U+c)(\frac{1}{2}F^2 + K^2) + (\beta - F^2U)\right]\Psi_2 = 0 \end{cases}$$

3 cases:

β=0 ⇒ (1)

β≠0 ⇒ (2)

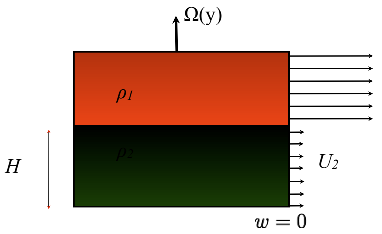
U=0 ⇒ (3)

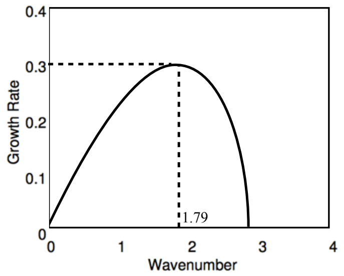
$$[A]\Psi_1 + [B]\Psi_2 = 0, \qquad [C]\Psi_1 + [D]\Psi_2 = 0$$

$$\Rightarrow \qquad AD - BC = 0$$

Dispersion relation $\beta=0$ (1)

$$c = \pm U \left[\frac{K^2 - F^2}{K^2 + F^2}\right]^{\frac{1}{2}} \qquad \sigma = Uk \left[\frac{F^2 - K^2}{K^2 + F^2}\right]^{\frac{1}{2}}$$





- Instability for all U
- Wave number cut-off at K>F=2.82/R_d
- No low wavenumber cut-off (k=2π/λ)
- Growth rate is maximum at 1.79/R_d

Dispersion relation $\beta \neq 0$ (2)

$$c = -\frac{\beta}{K^2+F^2}\left\{1 + \frac{F^2}{2K^2} \pm \frac{F^2}{2K^2}\left[1 + \frac{4K^4(K^4 - F^4)}{k_\beta^4 F^4}\right]^{\frac{1}{2}}\right\} \qquad k_\beta = \sqrt{\frac{\beta}{U}}$$

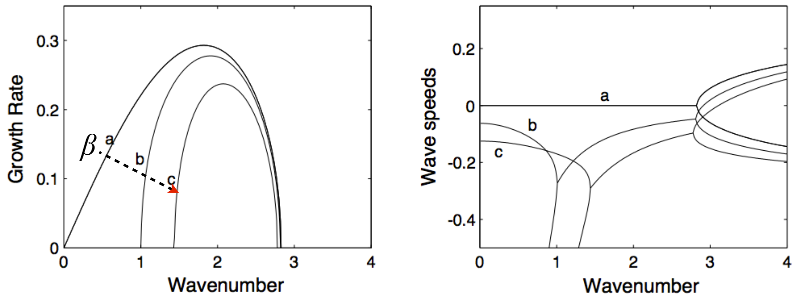


Fig. 6.14 Growth rates and wave speeds for the two-layer baroclinic instability problem, from (6.114), with three (nondimensional) values of β: a, $\gamma = 0$ ($k_\beta = 0$); b, $\gamma = 0.5$ ($\hat{k}_\beta = \sqrt{2}$); c, $\gamma = 1$ ($\hat{k}_\beta = 2$). As β increases, so does the low-wavenumber cut-off to instability, but the high-wavenumber cut-off is little changed. (The solutions are obtained from (6.114), with $\hat{k}_d = \sqrt{8}$ and $U_1 = -U_2 = 1/4$.)

from Vallis 2009

zero shear $U=0$ (3)

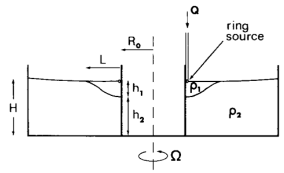
$$c = -\frac{\beta}{K^2+F^2}\left\{1 + \frac{F^2}{2K^2} \pm \frac{F^2}{2K^2}\left[1 + \frac{4K^4(K^4 - F^4)}{F_\beta^4 F^4}\right]^{\frac{1}{2}}\right\}$$

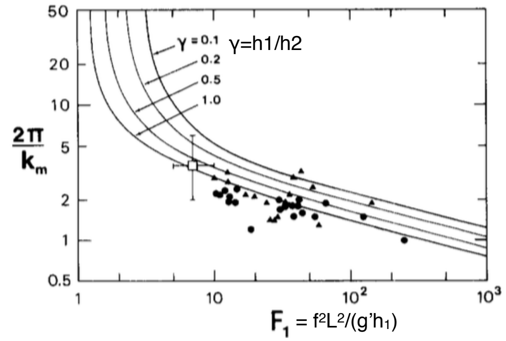
baroclinic mode
$\psi_1 + \psi_2 = 0$

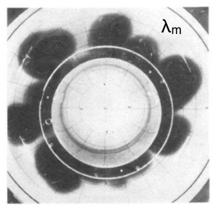
$$c = -\frac{\beta}{K^2+F^2}$$

in the barotropic mode: Rossby waves
$\psi_1 = \psi_2$

$$c = -\frac{\beta}{K^2}$$

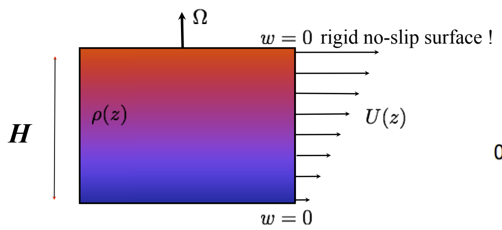






Griffiths & Linden Dyn. Atm. Oceans 1981

Eady problem (1949) $\beta = 0$: $f = f_0$, stratification N



Basic state :

$$fU = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$
$$0 = -\frac{\partial \bar{p}}{\partial z} - \bar{\rho} g \quad \Rightarrow$$
$$\frac{dU}{dz} = \frac{g}{f\rho_0}\frac{\partial \bar{\rho}}{\partial y}$$

SW-equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$
$$0 = -\frac{\partial p}{\partial z} - \rho g$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

perturbations:

$$\begin{aligned} u &= U(z) + u' \\ v &= v' \\ w &= w' \\ \rho &= \bar{\rho}(y,z) + \rho'(x,y,z) \\ p &= \bar{p}(y,z) + p'(x,y,z) \end{aligned}$$

Baroclinic instability (on the Eady problem)

Vorticity equation (after cross differentiation and use of continuity) :

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - (\omega + f)\frac{\partial w}{\partial z} = 0$$

with perturbations and linearized this becomes

$$\frac{\partial \omega'}{\partial t} + U\frac{\partial \omega'}{\partial x} - f\frac{\partial w'}{\partial z} = 0$$

for the density equation and hydrostatic balance for the perturbation

$$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + v'\frac{\partial \bar{\rho}}{\partial y} - \frac{\rho_0 N^2 w'}{g} = 0 \qquad \frac{\partial p'}{\partial z} + \rho' g = 0$$

derive an expression for w' $\Rightarrow$

Baroclinic instability (on the Eady problem)

$$w' = -\frac{1}{\rho N^2}\left[\left(\frac{\partial}{\partial z} + U\frac{\partial}{\partial x}\right)\frac{\partial p'}{\partial z} - \frac{dU}{dz}\frac{\partial p'}{\partial x}\right]$$

The pressure perturbation equation becomes:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla_H p' - \frac{f^2}{N^2}\frac{\partial^2 p'}{\partial z^2}\right] = 0$$

Consider a flow between z=0 and z=H, and
p'=P(z) exp(i[kx +ly -ωt])

P(z) is solved with Δp=0 and gives:
P= A cosh μ(z/H–1/2) + B sinh μ(z/H–1/2)
with $\mu^2 = (N/f)^2 (k^2+l^2) H = R_d^2 K^2$
w=0 at z=0,H ….

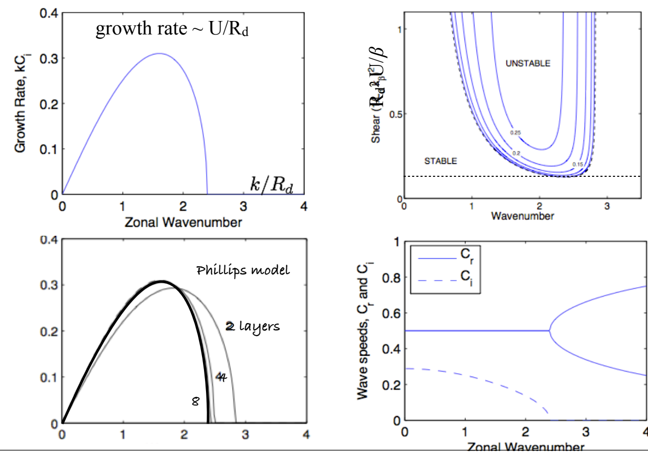
$$c = \frac{U_0}{2} \pm \frac{U_0}{\mu}\sqrt{\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\left(\frac{\mu}{2} - coth\frac{\mu}{2}\right)}$$

Baroclinic instability (on the Eady problem)

$$c = \frac{U_0}{2} \pm \frac{U_0}{\mu}\sqrt{\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\left(\frac{\mu}{2} - coth\frac{\mu}{2}\right)}$$

$C_i \neq 0$ => instability

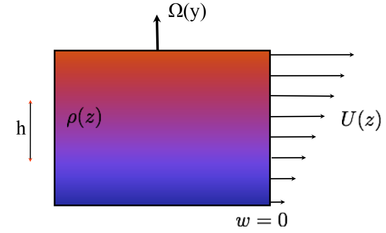
1/F=μ < 2.4 flow is unstable



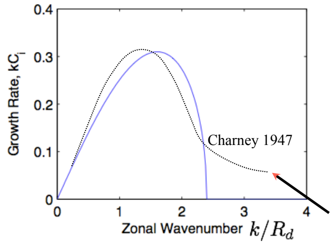
From Vallis 2009

minimum shear
$U_s/\beta > R_d/4$
for instability
= shear that changes
sign in PV in
the domain

Charney problem (1947) $\beta \neq 0$ basics



Vertical scale depends on vertical shear h=[(*f/N*)² *dU/dy*]/*β*



critical layer: U=c ...

No short wave cutoff due to Green's modes (Pedlosky 1986; Vallis 2006)

Baroclinic instability Quasi geostrophic equations

Stratified fluid and β≠0 (Charney 1947)

$$\frac{\partial}{\partial t}Q + \bar{u}.\nabla Q = 0 \qquad Q = \nabla^2\psi + \beta y + \frac{\partial}{\partial z}(S\frac{\partial \psi}{\partial z}) \qquad S = \frac{f_0^2}{N^2}$$

$$\frac{\partial b}{\partial t} + \bar{u}.\nabla b = 0 \qquad (\text{at } z = 0, H; \text{for } w = 0 \text{ and } b = f_0\frac{\psi}{\partial z})$$

perturbations of the form $\psi = Re\, A(y,z)\, e^{ik(x-ct)}$

$$(U-c)(\frac{\partial^2\tilde{\psi}}{\partial t^2} + \frac{\partial}{\partial z}(S\frac{\partial\tilde{\psi}}{\partial z}) - k^2\tilde{\psi}) + Q_y\tilde{\psi} = 0 \qquad 0 < z < H$$

$$(U-c)\frac{\partial\tilde{\psi}_z}{\partial z} - \frac{\partial\tilde{\psi}}{\partial z} = 0 \qquad z = 0, H$$

As before, integrate by parts and multiply by the complex conjugate to obtain the condition

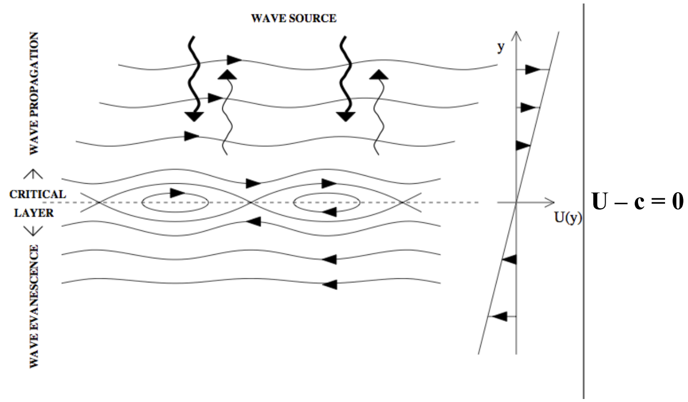
$$-c_i\int_{y_1}^{y_2}\left\{\int_0^H \frac{Q_y}{|U-c|^2}|\phi|^2 dz + \left[\frac{SU_z|\tilde{\psi}|^2}{|U-c|}\right]_0^H\right\} dy = 0$$

critical layer when U – c = 0 ==>

$$-c_i\int_{y_1}^{y_2}\left\{\int_0^H \frac{Q_y}{|U-c|^2}|\phi|^2 dz + \left[\frac{SU_z|\tilde{\psi}|^2}{|U-c|}\right]_0^H\right\} dy = 0$$

Instability if

Q_y changes sign,
Q_y is the opposite or the same sign of U_z at resp. z=0,H
Q_y =0 and U_z has the same sign at the two boundaries



Potential vorticity equation

HOMOGENEOUS FLUID Intermezzo

Cross differentiation and substitution of the continuity in the Euler relations directly gives for the vorticity $\bar{\omega} = (0, 0, \omega_z)$, with $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$:

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} + f(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

with

$$\frac{Dh}{Dt} + h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0.$$

This gives the conservation of potential vorticity :

$$\frac{D\omega_z}{Dt} = \frac{\omega_z + f}{h}\frac{Dh}{Dt}$$

or

$$\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0 \quad = \frac{D}{Dt}Q_{homogeneous}$$

Arnold-Blumen theorem

Consider the potential vorticity equation

$$\frac{\partial}{\partial t}Q + \bar{u}.\nabla Q = 0 \qquad Q = \nabla^2\psi + \beta y + \frac{\partial}{\partial z}(S\frac{\partial \psi}{\partial z}) \qquad S = \frac{f_0^2}{N^2}$$

with J(a,b) the Jacobian

$$\frac{\partial}{\partial t}Q + J(\psi, Q) = 0$$

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$$

For stationary flows $J(a,b)=0$, i.e there is function f with $Q=f(\psi)$.
These can be coherent structures, stationary flow platterns etc.
If the structure has a uniform translation velocity c, then we have with respect to the moving frame of reference $\psi = \psi(x - ct, y)$

and functional f $\qquad Q = f(\psi + cy) \qquad$ f stable or unstable?

$Q = f(\psi + cy)$ $\qquad$ f stable or unstable?

Perturbation of ψ

$$\psi(x, y, t) = \bar{\psi}(x - ct, y) + \phi(x - ct, y, t)$$

relation for the perturbation ϕ

$$\frac{\partial}{\partial t}(\nabla^2\phi - S\phi) + J(\bar{\psi} + cy, \nabla^2\phi - S\phi - f'\phi) + J(\phi, \nabla^2\phi) = 0$$

Consider the formal stability of f (note $f' = \partial f/\partial\psi = \partial Q/\partial\psi$) multiply with $-\phi$ integrate over the domain, suppose the domain is closed

$$\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}\left[(\nabla\phi)^2 + S\phi^2\right] d\Omega + \int_\Omega \nabla^2\phi \; J(\bar{\psi} + cy, \phi)\, d\Omega = 0$$

= total (kinetic + potential energy) + ~ total enstrophy
using continuity and boundary conditions, multiplication with $(\nabla^2\phi - S\phi^2)/f'$ and integrating…(see e.g Pedlosky, Springer 1987)

$$\frac{\partial}{\partial t}L(\phi) = 0 \qquad L(\phi) = \frac{1}{2}\int_\Omega \left[(\nabla\phi)^2 + S\phi^2 + \frac{1}{f'}(\nabla^2\phi - S\phi)^2\right] d\Omega$$

$$\frac{\partial}{\partial t}L(\phi) = 0 \qquad L(\phi) = \frac{1}{2}\int_\Omega \left[(\nabla\phi)^2 + S\phi^2 + \frac{1}{f'}(\nabla^2\phi - S\phi)^2\right] d\Omega$$

if $L(\phi)$ definite positive/negative

=> $L \sim ||\phi||^2$ = conserved in time

sign of f' determines stability, i.e. increase or decrease in energy of the system

$Q = \boldsymbol{f}(\psi + cy)$ ***f*** stable or unstable?

The same result can be obtained after transformation to the co-moving frame of reference $\phi(x-ct,y,t) = \hat{\phi}(t)\chi(x-ct,y)$

For stability:

$$\boldsymbol{f}' = \frac{\partial \boldsymbol{f}(\psi + cy)}{\partial \psi} > 0$$

This is a *sufficient condition* and its violation is a necessary condition for instability.

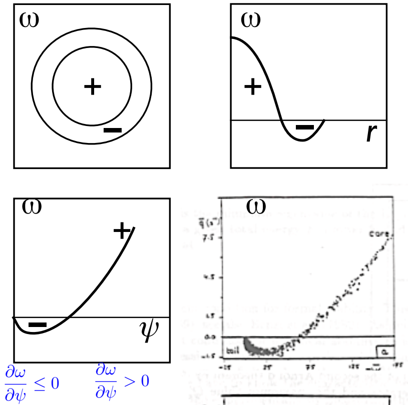
In practice this means after a transformation to the co-moving frame of reference (x-*ct*) for stability

Formal stability
stronger than normal mode

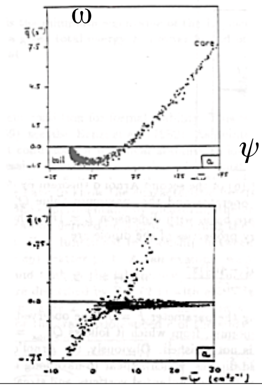
$$\frac{\partial \omega}{\partial \psi} > 0$$

Maxwell 1855 (letter to Lord Kelvin alias W. Thomson)
Arnold 1966a,1966b
Blumen (JAS) 1968
Pedlosky (book) 1987
Ripa JFM 1991, 1993

Example (shown before)



$\frac{\partial \omega}{\partial \psi} \le 0$ $\frac{\partial \omega}{\partial \psi} > 0$



ring with ω<0
is unstable

Weakness of linear theory

- Only small perturbations
- amplitude grows in time so that nonlinear effect become gradually more important
- Final state is unknown

Advantage:

- simple technique
- mode decomposition is physically comprehensible
- gives a first understanding about hydrodynamics instability

Non-linear methods and weakly nonlinear methods:
see
- Godreche and Manneville (Saclay 1990)
- Drazin and Reid Ch 7
- Manneville 1990
- Guyon et al 2001

....

Different concepts of instability (Holm et al Phys lett. 1985)

- spectral instability (normal modes)
- linear instability
- formal instability
- nonlinear instability

Linearization and infinitesimal perturbations

consider a dynamical system
d*u*/dt=*X*(u)

Spectrally stable:
For a dynamical system *u*= d*u*/dt= X(*u*), an equilibrium point U_e satisfying $X(U_e) = 0$ is called spectrally stable, provided the spectrum of the linearized operator $DX(U_e)$ has no strictly positive real part.

Linearized stability
The equilibrium solution u~is called linearized stable or linearly stable relative to a norm II~uII on infinitesimal variations~iu provided for every s>0 there is a 3>0 such that if I!~uI<l3att= 0,then II~uI<Iefort>0,where~uevolvesaccordingto(au)= DX(Ue)~ ~U.

Lyapunov stable… (nonlinear instability)