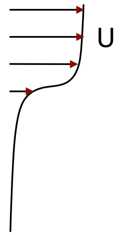
Kelvin Helmholtz instability
Hölmböe instability
Rayleigh-Taylor instability

Methods: normal mode instability
Energy of particles
(heuristic method)

Kelvin Helmholtz in the Atmosphere

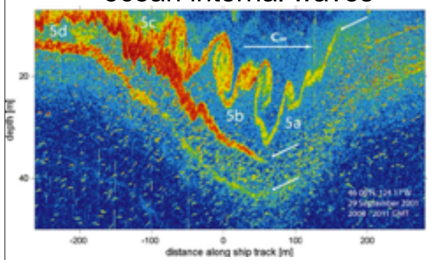


ocean wind waves

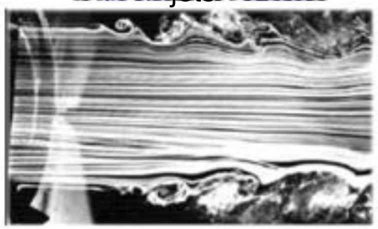
shear regions

Rio Nero and Amazone river waters
rencontre entre

ocean internal waves

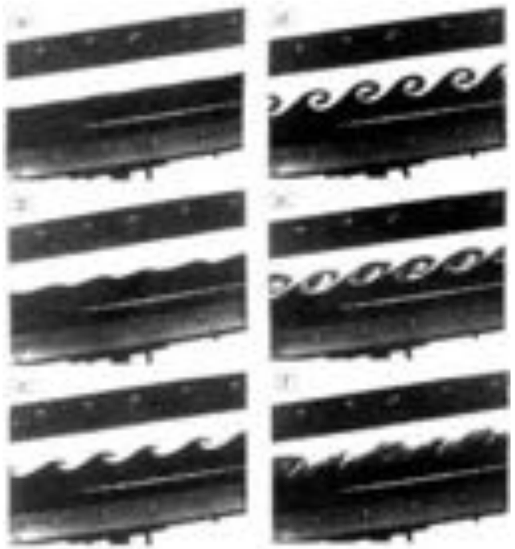


jets



IN THE LABORATORY

Time



- Constant wavelength
- Amplitude increase
- reaches a maximum (saturation)
- turbulence

Linear stability analyses example

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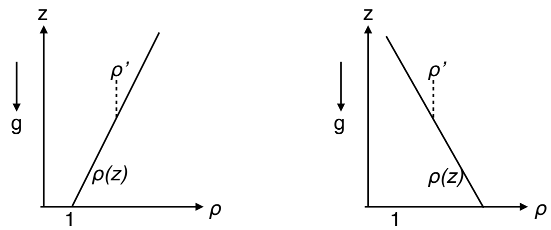
Kelvin Helmholtz (Thorpe 1969)

normal mode method
(KH homogeneous fluid)

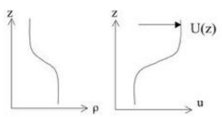
(INSTABILITY WHEN $\Delta E_k > W$)

($\Delta E_p<0$ $\Delta E_p > 0$)

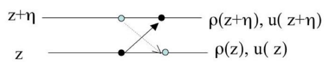


stability of particles in a stratified fluid

Stratified shear flows and instability



Consider the *exchange* of a fluid parcel with one at another level in a stably stratified fluid.*



How much work W is being done, and how much energy is made free ? (Consider the leading order density effects).

*Suppose $u(z+\eta)=u+\delta u$, and after exchange $u=u_{mean}=(u+(u+\delta u))/2$
Inertia effects are negligible on density, i.e. $\rho=\rho_0$ *(Boussinesq approximation)*

$$KE_1 = \frac{\rho_0}{2}[u^2 + (u+\delta u)^2] = \frac{\rho_0}{2}[2u^2 + 2u\delta u + (\delta u)^2]$$

After exchange of the two particles :

$$KE_2 = \frac{\rho_0}{2}[2(\frac{u+(u+\delta u)}{2})^2] = \frac{\rho_0}{2}[2u^2 + 2u\delta u + 1/2(\delta u)^2]$$

$$\Delta KE = KE_2 - KE_1 = -\frac{\rho_0}{4}(\delta u)^2$$

The change in buoyancy is

$$\Delta B = g\rho(z) - g\rho(z+\eta) = g\rho(z) - g[\rho(z) + \eta\frac{d\rho}{dz} + ...] \approx -g\frac{d\rho}{dz}\eta$$

with $\rho(z) = \rho(z_0) + \frac{\rho_0}{dz}(z - z_0) + ... \approx \rho(z_0)$ and the work on a single particle at the level δz is thus

$$W = \int_0^{\delta z} -g\frac{d\rho}{dz}\eta d\eta = -g\frac{d\rho}{dz}\frac{(\delta z)^2}{2},$$

The work for the exchange is then : $W = -g\frac{d\rho}{dz}(\delta z)^2$.

There is instability when $\Delta KE > W$, or

$$\frac{\rho_0}{4}(\delta u)^2 > -g\frac{d\rho}{dz}(\delta z)^2$$

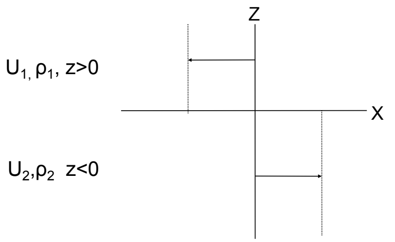
with

$$Ri = \frac{-\frac{g}{\rho}\frac{d\rho}{dz}}{(\frac{du}{dz})^2} < \frac{1}{4}$$

This is the Richardson criterion for Kelvin Helmholtz instability.

Instability of a vortex sheet

using Bernoulli



δρ = 0, ρ₁ = ρ₂, Incompressible flow.

$$U_{1,2} = \frac{(U_1 + U_2)}{2} \pm \frac{U_1 - U_2}{2} = C \pm \frac{U}{2}$$

The *frame is moving with speed* C (so that $U_i = \pm U/2$)

The ***basic flow*** represents a vorticity sheet generated by two parallel flows, of which the instability is driven by inertial forces.

Linear stability analyses: perturbation of this basic flow –>

Define in each layer a **velocity potential** $\mathbf{u}_i$ **=** grad φ_i, so that

$$U_1 = \frac{\partial \phi_1}{\partial x} \qquad U_2 = \frac{\partial \phi_2}{\partial x}$$

			by continuity	
with	φ_1	above the interface	$\Delta\varphi_1=0$	($z>\zeta$)
and	φ_2	below the interface	$\Delta\varphi_2=0$	($z<\zeta$)

Since we consider potential flows above and below the interface, we may use ***Bernoulli*** for this potential flow

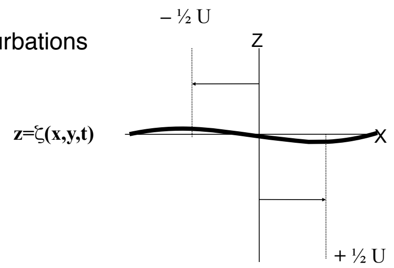
(substitute $u=\nabla\varphi$ in the Euler equations, and note that $u \times \omega=0$)

For the basic flow $\frac{1}{2}U^2 + gz + \int \frac{\nabla p}{\rho} = H = \text{constant along streamlines}$

But since perturbations depend on time, we must use

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}U^2 + gz + \frac{P}{\rho} = H \text{ with } U = \nabla\phi$$

The perturbations



at the level z=ζ(x,y,t), that is the interface, we have:

Just above: **z > ζ** : $\phi_1 = -\frac{1}{2} U x + \phi'_1$ (= basic flow + perturbation of O(ε))

Just below: z < ζ : $\phi_2 = \frac{1}{2} U x + \phi'_2$.

+ Boundary conditions….

interface and flow at infinity —->

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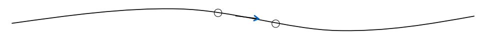
Interface conditions:

See Drazin and Reid page 16-22

We follow the Langrangian motion of a particle near the interface



1: Cinematic boundary condition *imposes continuity of displacements at the interface we take the* **total derivative D/Dt= ∂ₜ +u.∇**)

I
$$w_1 = \frac{\partial \phi'_1}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \left(-\frac{1}{2}U + \frac{\partial \phi'_1}{\partial x}\right)_{z=\zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial \phi'_1}{\partial z}\frac{\partial \zeta}{\partial z} \qquad z>\zeta$$
$$= \frac{\partial \zeta}{\partial t} + \left(-\frac{1}{2}U + u_1\right)_{z=\zeta} \frac{\partial \zeta}{\partial x} + (w_1)_{z=\zeta}\frac{\partial \zeta}{\partial z}$$

II
$$w_2 = \frac{\partial \phi'_2}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \left(\frac{1}{2}U + \frac{\partial \phi'_2}{\partial x}\right)_{z=\zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial \phi'_2}{\partial z}\frac{\partial \zeta}{\partial z} \qquad z<\zeta$$
$$= \frac{\partial \zeta}{\partial t} + \left(\frac{1}{2}U + u_2\right)_{z=\zeta} \frac{\partial \zeta}{\partial x} + (w_2)_{z=\zeta}\frac{\partial \zeta}{\partial z}$$

In *linear approximation* (with z and primes of O(ε))

I
$$w_1 = \frac{\partial \phi'_1}{\partial z} = \frac{\partial \zeta}{\partial t} - \frac{1}{2}U\frac{\partial \zeta}{\partial x}$$

II
$$w_2 = \frac{\partial \phi'_2}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{1}{2}U\frac{\partial \zeta}{\partial x}$$

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2: Dynamics boundary condition

Continuity of pressure across the vortex sheet

In Bernoulli
$$\frac{\partial \phi_i}{\partial t} + \frac{1}{2}(\nabla\phi_i)^2 + gz + \frac{P_i}{\rho} = H$$
$$\text{with } \nabla\phi_1 = -\frac{1}{2}U + \frac{\partial \phi'_1}{\partial x} \text{ and } \nabla\phi_2 = \frac{1}{2}U + \frac{\partial \phi'_2}{\partial x}$$

continuity of pressure $(P_1 - P_2)_{z=\zeta} = 0$

We obtain after linearisation :

III
$$\left(\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1}{\partial t}\right)_{z=0} = \frac{1}{2}U\left(\frac{\partial \phi'_2}{\partial x} + \frac{\partial \phi'_1}{\partial x}\right)_{z=0}$$

I,II,III are linear and can be solved if we represent the sheet displacement

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Consider **perturbations** of the form

$\varphi'_1, \varphi'_2 = F(z)\ e^{i(kx)+\sigma t}$ and $\zeta = A\ e^{i(kx)+\sigma t}$

These are Fourier components or normal modes! What is F(z) ?

Condition at infinity: the amplitude of the perturbations goes to zero!

Since $\Delta\varphi'_i=0$ $\varphi'_i = B_1 e^{-kz} + B_2 e^{kz}$

$\varphi'_i \to 0$ for $z \to +\infty$ thus for **z>0** **$B_2=0$**
$\varphi'_i \to 0$ for $z \to -\infty$ thus for **z<0** **$B_1=0$**

We can now **solve the form of ζ*, φ*₁, φ*₂ with amplitudes A, B₁, and B₂**

$\zeta = \mathbf{A}\ e^{ikx+\sigma t}$,

$\varphi'_1 = B_1 e^{-kz} e^{ikx+\sigma t}$. $\varphi'_2 = B_2 e^{kz} e^{ikx+\sigma t}$

Substitution in conditions I and II:

$-kB_1 = (\sigma - \frac{1}{2}i\,k\,U)\,A$
$-kB_2 = (\sigma + \frac{1}{2}i\,k\,U)\,A$

and condition III: $i\,[\sigma(B_2 - B_1)_{z=0} + \frac{1}{2}U(B_2k + B_1k)_{z=0}]\,e^{i(kx)} = H$

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With ***Im(H)=0*** we obtain:

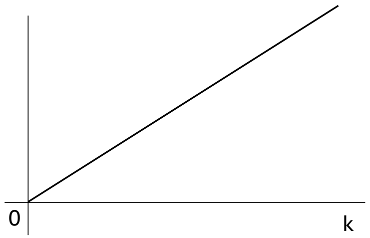
$\sigma = \frac{1}{2} ik(U_1+U_2) \pm \frac{1}{2} k(U_1 - U_2)$

for $U_1 = -U_2$ this reduces to

σ=±kU

- exponential growth for any velocity for σ>0
- growth rate depends on U

σ= kU



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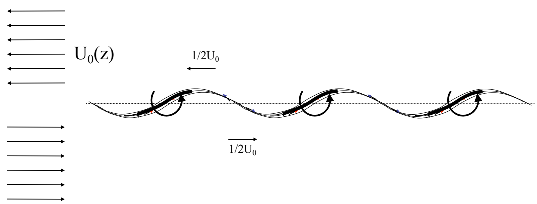
σ=±kU

σ(k) is the dispersion relation showing the variation of growth rate with k. For σ>0, k≠0 the sheet is unstable. Small wavelengths grow faster than short ones.

All wave lengths are unstable no matter how small U is!. In reality often there is a cutoff for small wavelengths as we will see later.



(Vertical velocity is out phase with pressure and velocity, and horizontal vorticity $\omega_y \approx -\frac{\partial w}{\partial x} = -ikw$)

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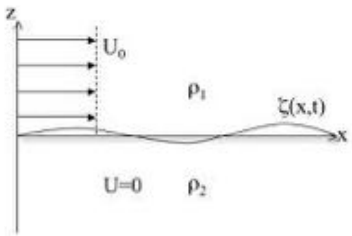
Laminar basic flow ; with layers 1 and 2 of different density



Viscous effets are considered negligible and the fluid is incompressible.
This flow satisfies the Euler equations, continuity and hydrostatic balance.

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Perturbations

The basic flow satisfies
The Euler equations, continuity and hydrostatic balance are ;

$$\frac{\partial \vec{u}}{\partial t} + u.\nabla\vec{u} = -\frac{\nabla p}{\rho} - g \qquad \nabla.\vec{u} = 0 \qquad \frac{dp}{dz} = -\rho g$$

We suppose a perturbation of the form

$$p = P + p'$$
$$\rho = \rho_i + \rho_i' \quad \text{(i=1,2)}$$
$$u_1 = U_0 + u_1'$$
$$u_2 = u_2'$$

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The basic flow is given by

$$(u_1, w_1) = (U_0, 0) \qquad (u_2, w_2) = (0, 0)$$
$$p(z) = P - \rho_1 g z \;\; (z > 0) \qquad p(z) = P - \rho_2 g z \;\; (z < 0)$$

Substitue the perturbations (neglect second order terms), so that we obtain :

$$\nabla.(\bar{U}_0 + \bar{u}') = 0$$
$$\frac{\partial \bar{U}_0 + \bar{u}'}{\partial t} + (\bar{U}_0 + \bar{u}')\frac{\partial(\bar{U}_0 + \bar{u}')}{\partial x} = \frac{\nabla(P + p')}{\rho_0 + \rho'}$$
$$\Longrightarrow$$

For the upper layer we obtain : Lower layer:

$$\frac{\partial u_i'}{\partial x} + \frac{\partial w_i'}{\partial z} = 0 \quad (i = 1, 2) \qquad (1)$$
$$\frac{\partial u_1'}{\partial t} + U_0 \frac{\partial u_1'}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1'}{\partial x} \qquad \Bigg| \qquad \frac{\partial u_2'}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial x} \qquad (2)$$
$$\frac{\partial w_1'}{\partial t} + U_0 \frac{\partial w_1'}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1'}{\partial z} \qquad \Bigg| \qquad \frac{\partial w_2'}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial z} \qquad (3)$$

We use perturbations of the form

$$(u', w', p', \zeta') = (\hat{u}, \hat{w}, \hat{p}, \hat{\zeta})(z)e^{ikx - i\omega t}$$

The fonction $(\hat{u}, \hat{w}, \hat{p}, \hat{\zeta})(z)$ can be derived from eqs. (1,2 and 3).
With $\frac{\partial(2)}{\partial x} + \frac{\partial(3)}{\partial z} = -\nabla^2 p_i'$ and continuity one obtains $\nabla^2 p_i' = 0$.
Using the expression for the perturbations above yields

$$\frac{\partial^2 p_i'}{\partial z^2} - k^2 p_i' = 0,$$

with solutions $p_i' = A_i e^{kz} + B_i e^{-kz}$.
Under the condition that perturbations disappear with distance from the interface $z \to \pm\infty$ $\hat{p'} \to 0$ we obtain

In layer 1 : $(u', w', p', \zeta')_1 \sim e^{-kz} e^{i(kx - \omega t)}$

In layer 2 : $(u', w', p', \zeta')_2 \sim e^{kz} e^{i(kx - \omega t)}$

Note : The basic equations provide information about the phase of the pressure with respect to the vertical motion. Substitution of the perturbations in the latter equation shows (omitting primes)

$$-i(\omega - kU_0)w_1 = -\frac{k}{\rho}p_1$$
$$-i\omega w_2 = -\frac{k}{\rho}p_2$$

(Vertical velocity is out phase with pressure and velocity, and horizontal vorticity $\omega_y \approx -\frac{\partial w}{\partial x} = -ikw$)

Interface conditions

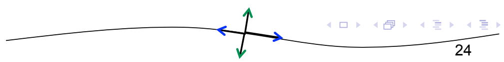
Lagrangian motion of a particle at the interface

I) Kinematic interface condition : particles remain at the interface. Consider a particle at the interface $\zeta(x, t)$, given by $z = \zeta(x, t)$. By continuity, the vertical motion of this particle should match the velocity above and below the interface :

$$\text{upper layer} \quad \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + U_0 \frac{\partial \zeta}{\partial x} = w_1$$
$$\text{lower layer} \quad \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} = w_2$$

II) Dynamic condition : continuity of forces across the interface. Here, normal to the interface, pressure and gravity

$$p_1 - p_2 = (\rho_1 - \rho_2)g\zeta \quad \text{for } z = 0$$



force balance **normal** to the interface

We consider the motion in the vertical direction :

$$\frac{\partial \zeta}{\partial t} + U_0 \frac{\partial \zeta}{\partial x} = w_1$$
$$\frac{\partial \zeta}{\partial t} = w_2$$
$$p_1 - p_2 = (\rho_1 - \rho_2) g \zeta$$
$$\frac{\partial w_2'}{\partial t} = -\frac{1}{\rho_2} \frac{\partial p_2'}{\partial z}$$
$$\frac{\partial w_1'}{\partial t} + U_0 \frac{\partial w_1'}{\partial x} = -\frac{1}{\rho_1} \frac{\partial p_1'}{\partial z}$$

Substitute the perturbations and write in matrix form to determine the dispersion relation.

$$i(kU_0 - \omega)\zeta - W_1 = 0$$
$$-i\omega\zeta - W_2 = 0$$
$$g(\rho_2 - \rho_1)\zeta + P_1 - P_2 = 0$$
$$-i\omega W_2 + \frac{k}{\rho_2} P_2 = 0$$
$$i(kU_0 - \omega) W_1 + \frac{k}{\rho_1} P_1 = 0$$

Elimination of W_1, W_2 and P_1, P_2 provides an equation in ζ

Solution

Sometimes it is easier to write this in the form of a matrix

$$\begin{pmatrix} i(kU_0-\omega) & -1 & 0 & 0 & 0 \\ -i\omega & 0 & -1 & 0 & 0 \\ g(\rho_2-\rho_1) & 0 & 0 & 1 & -1 \\ 0 & 0 & -i\omega & 0 & k/\rho_2 \\ 0 & i(kU_0-\omega) & 0 & k/\rho_1 & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ W_1 \\ W_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

If Det=0 then nontrivial solution exist. If there are many equations make use of a program like Python, Maple, Scylab, Matlab, or Mathematica. This provides the dispersion relation $\omega(k)$:

$$(\rho_1 + \rho_2)\omega^2 - 2kU_0\rho_1\omega + k^2 U_0^2 \rho_1 - kg(\rho_2 - \rho_1) = 0.$$

Interpretation 1

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2 U_0^2 \rho_1 \rho_2 - kg(\rho_2 - \rho_1)(\rho_2 + \rho_1)}}{(\rho_1 + \rho_2)}$$

Remind that the form of the perturbation is $\sim e^{i(kx - \omega t)}$

- Water-Air interface : $U_0 = 0$ et $\rho_1 = 0$

  From the dispersion relation we obtain $\mathrm{Im}(\omega) = 0$, and $\mathrm{Re}(\omega)$:

  $$\omega = \pm\sqrt{kg}$$

  $\omega_i = 0 \rightarrow e^{\omega_i t} = 1 \rightarrow$ stable.

  $\omega_r \neq 0 \rightarrow$ surface waves with phase velocity : $c = \sqrt{g/k}$.

Interpretation 2

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1)}}{(\rho_1+\rho_2)}$$

- Stable fluid interface, but without shear
  i.e. $U_0 = 0$ and $\rho_1 > 0$

The dispersion relation reduces to (only Re(ω)$\neq$ 0) :

$$\omega = \pm\sqrt{\frac{kg(\rho_2-\rho_1)}{(\rho_1+\rho_2)}}$$

$\rho_1 < \rho_2 \rightarrow \omega_i = 0$ stable ($\rho_1 > \rho_2$ instable)
The phase velocity is for interfacial gravity waves :

$$c = \pm\sqrt{\frac{g}{k}\frac{(\rho_2-\rho_1)}{(\rho_1+\rho_2)}} \qquad =\pm\sqrt{g'/k}$$

Interpretation 3

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1)}}{(\rho_1+\rho_2)}$$

- Stable density interface with shear $\rho_1 \neq \rho_2$, $U_0 \neq 0$, $\rho_2 > \rho_1$

There is stability when :

$$U_0^2 \leq \frac{g}{|k|\rho_1\rho_2}(\rho_2^2 - \rho_1^2)$$

There is instability when $\pm\omega_i \neq 0$, i.e. for

$$4k^2U_0^2\frac{\rho_1\rho_2}{\bar\rho^2} - 2kg\frac{\Delta\rho}{\bar\rho} \approx 2k(2kU_0^2 - g\frac{\Delta\rho}{\bar\rho}) > 0$$

From this expression, derive instability for (a typical length scale L).

$$Ri = \frac{-g}{\rho_0}\frac{\frac{d\rho}{dz}}{(\frac{du}{dz})^2} < \frac{1}{4}$$

EXERCISE

Modify the dispersion relation for a surface tension T.
(note that we only consider the force perpendicular
to the interface and not the forces tangential to the interface)

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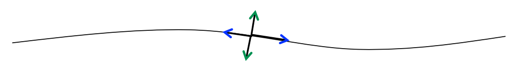

La condition de pression linéarisé à l'interface donne

$$p_2 - p_1 = (\rho_2-\rho_1)g\zeta - T\frac{d^2\zeta}{dz^2} \quad \text{pour } z = 0$$

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1) - k^3T}}{(\rho_1+\rho_2)}$$

$e^{-i\omega t}$

growth $Re(-i\omega t) > 0$

waves $Im(-i\omega t) \neq 0$

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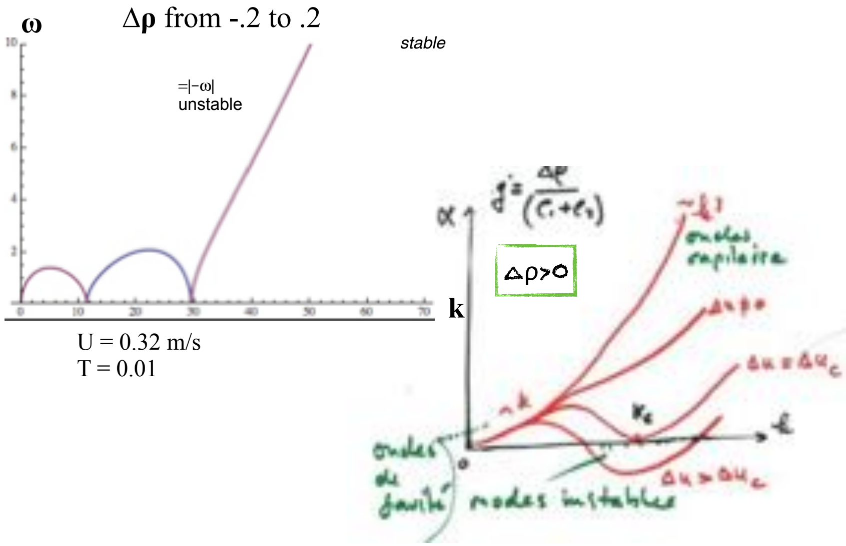
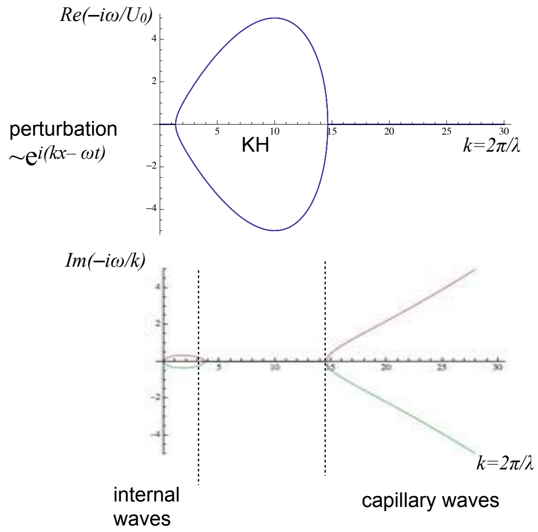
$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1) - k^3T}}{(\rho_1+\rho_2)}$$



Graphical representation

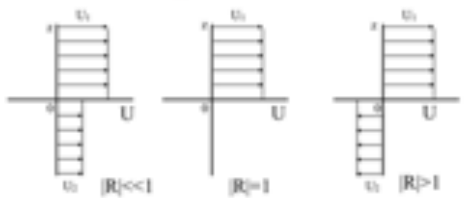
Δρ>0, stable stratification

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1) - k^3T}}{(\rho_1+\rho_2)}$$



comments

Effect of velocity ratio $R = \frac{\Delta U}{2\bar{U}}$ on vortex sheet profile



R=ΔU/2U

convection speed $(U_1 + U_2)/2$

growth $\Delta U = (U_1 - U_2)$

The velocity ratio R is important to the nature of the instability :

For KH flow the interface conditions (pressure / continuity w) impose the dispersion relation :

$$(c - U_1)^2 + (c - U_2)^2 = 0$$

with solution $c \equiv \frac{\omega}{k} = \bar{U} \pm i\frac{\Delta U}{2} = \bar{U}(1 \pm iR)$ and $c_r = \bar{U}$ the propagation speed of the waves.