Kelvin Helmholtz instability
Hölmböe instability
Rayleigh-Taylor instability

Methods: normal mode instability
Energy of particles
(heuristic method)

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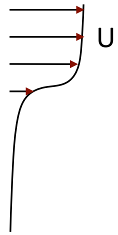
Kelvin Helmholtz in the Atmosphere



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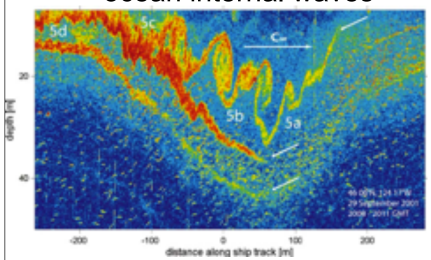
ocean wind waves

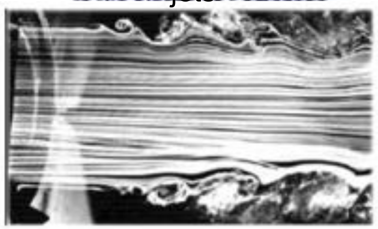
shear regions

Rio Nero and Amazone river waters
rencontre entre

ocean internal waves



jets



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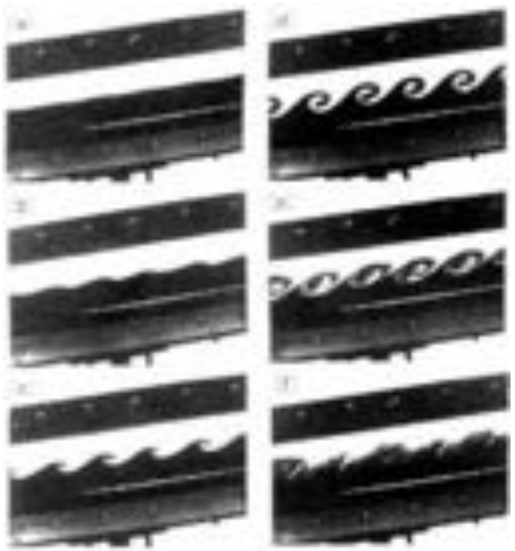
IN THE LABORATORY

Time



- Constant wavelength
- Amplitude increase
- reaches a maximum (saturation)
- turbulence

Linear stability analyses example

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Kelvin Helmholtz (Thorpe 1969)

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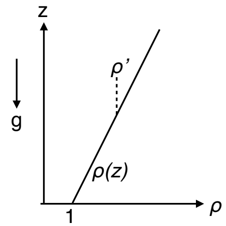
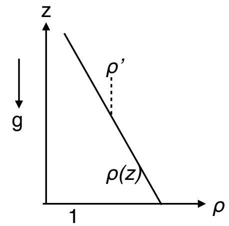
normal mode method
(KH homogeneous fluid)

(INSTABILITY WHEN ΔEk > W)

($\Delta E_p<0$ $\Delta E_p > 0$)





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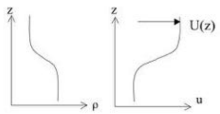
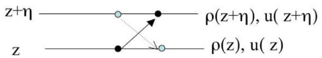
stability of particles in a stratified fluid

Stratified shear flows and instability



Consider the *exchange* of a fluid parcel with one at another level in a stably stratified fluid.*



How much work W is being done, and how much energy is made free ? (Consider the leading order density effects).

*Suppose $u(z+\eta)=u+\delta u$, and after exchange $u=u_{mean}=(u+(u+\delta u))/2$
Inertia effects are negligible on density, i.e. $\rho=\rho_0$ *(Boussinesq approximation)*

$$KE_1 = \frac{\rho_0}{2}[u^2 + (u+\delta u)^2] = \frac{\rho_0}{2}[2u^2 + 2u\delta u + (\delta u)^2]$$

After exchange of the two particles :

$$KE_2 = \frac{\rho_0}{2}[2(\frac{u+(u+\delta u)}{2})^2] = \frac{\rho_0}{2}[2u^2 + 2u\delta u + 1/2(\delta u)^2]$$

$$\Delta KE = KE_2 - KE_1 = -\frac{\rho_0}{4}(\delta u)^2$$

The change in buoyancy is

$$\Delta B = g\rho(z) - g\rho(z+\eta) = g\rho(z) - g[\rho(z) + \eta\frac{d\rho}{dz} + ...] \approx -g\frac{d\rho}{dz}\eta$$

with $\rho(z) = \rho(z_0) + \frac{\rho_0}{dz}(z - z_0) + ... \approx \rho(z_0)$ and the work on a single particle at the level δz is thus

$$W = \int_0^{\delta z} -g\frac{d\rho}{dz}\eta d\eta = -g\frac{d\rho}{dz}\frac{(\delta z)^2}{2},$$

The work for the exchange is then : $W = -g\frac{d\rho}{dz}(\delta z)^2$.

There is instability when $\Delta KE > W$, or

$$\frac{\rho_0}{4}(\delta u)^2 > -g\frac{d\rho}{dz}(\delta z)^2$$

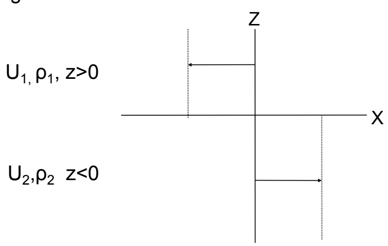
with

$$Ri = \frac{-\frac{g}{\rho}\frac{d\rho}{dz}}{(\frac{du}{dz})^2} < \frac{1}{4}$$

This is the Richardson criterion for Kelvin Helmholtz instability.

Instability of a vortex sheet

using Bernoulli



δρ = 0, ρ₁ = ρ₂, Incompressible flow.

$$U_{1,2} = \frac{(U_1 + U_2)}{2} \pm \frac{U_1 - U_2}{2} = C \pm \frac{U}{2}$$

The *frame is moving with speed* C (so that U_i= ±U/2)

The ***basic flow*** represents a vorticity sheet generated by two parallel flows, of which the instability is driven by inertial forces.

Linear stability analyses: perturbation of this basic flow –>

Define in each layer a **velocity potential u$_i$ =** grad φ_i, so that

$$U_1 = \frac{\partial \phi_1}{\partial x} \qquad U_2 = \frac{\partial \phi_2}{\partial x}$$

			by continuity	
with	φ_1	above the interface	$\Delta\varphi_1$=0	(z>ζ)
and	φ_2	below the interface	$\Delta\varphi_2$=0	(z<ζ)

Since we consider potential flows above and below the interface, we may use ***Bernoulli*** for this potential flow

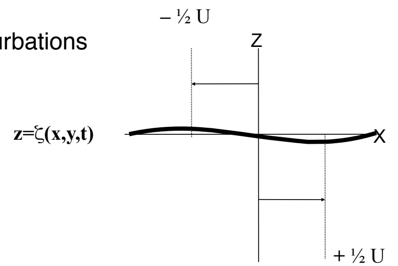
(substitute *u*=∇φ in the Euler equations, and note that *u* x *ω=0*)

For the basic flow $\frac{1}{2}U^2 + gz + \int \frac{\nabla p}{\rho} = H = \text{constant along streamlines}$

But since perturbations depend on time, we must use

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}U^2 + gz + \frac{P}{\rho} = H \text{ with } U = \nabla\phi$$

The perturbations



at the level z=ζ(x,y,t), that is the interface, we have:

Just above: **z > ζ** : $\phi_1 = -\frac{1}{2} U x + \phi'_1$ (= basic flow + perturbation of O(ε))

Just below: z < ζ : $\phi_2 = \frac{1}{2} U x + \phi'_2$.

+ Boundary conditions….

interface and flow at infinity —->

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Interface conditions:

See Drazin and Reid page 16-22

We follow the Langrangian motion of a particle near the interface

1: Cinematic boundary condition *imposes continuity of displacements at the interface we take the* **total derivative D/Dt= ∂_t +u.∇***)*

I $$w_1 = \frac{\partial\phi_1'}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + \left(-\frac{1}{2}U + \frac{\partial\phi_1'}{\partial x}\right)_{z=\zeta}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_1'}{\partial z}\frac{\partial\zeta}{\partial z}$$ z>ζ

$$= \frac{\partial\zeta}{\partial t} + \left(-\frac{1}{2}U + u_1\right)_{z=\zeta}\frac{\partial\zeta}{\partial x} + (w_1)_{z=\zeta}\frac{\partial\zeta}{\partial z}$$

II $$w_2 = \frac{\partial\phi_2'}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + \left(\frac{1}{2}U + \frac{\partial\phi_2'}{\partial x}\right)_{z=\zeta}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_2'}{\partial z}\frac{\partial\zeta}{\partial z}$$ z<ζ

$$= \frac{\partial\zeta}{\partial t} + \left(\frac{1}{2}U + u_2\right)_{z=\zeta}\frac{\partial\zeta}{\partial x} + (w_2)_{z=\zeta}\frac{\partial\zeta}{\partial z}$$

In *linear approximation* (with z and primes of O(ε))

I $$w_1 = \frac{\partial\phi_1'}{\partial z} = \frac{\partial\zeta}{\partial t} - \frac{1}{2}U\frac{\partial\zeta}{\partial x}$$

II $$w_2 = \frac{\partial\phi_2'}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{1}{2}U\frac{\partial\zeta}{\partial x}$$

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2: Dynamics boundary condition
Continuity of pressure across the vortex sheet

In Bernoulli $$\frac{\partial\phi_i}{\partial t} + \frac{1}{2}(\nabla\phi_i)^2 + gz + \frac{P_i}{\rho} = H$$

$$\text{with } \nabla\phi_1 = -\frac{1}{2}U + \frac{\partial\phi_1'}{\partial x} \text{ and } \nabla\phi_2 = \frac{1}{2}U + \frac{\partial\phi_2'}{\partial x}$$

continuity of pressure $(P_1 - P_2)_{z=\zeta} = 0$

We obtain after linearisation :

III $$\left(\frac{\partial\phi_2}{\partial t} - \frac{\partial\phi_1}{\partial t}\right)_{z=0} = \frac{1}{2}U\left(\frac{\partial\phi_2'}{\partial x} + \frac{\partial\phi_1'}{\partial x}\right)_{z=0}$$

I,II,III are linear and can be solved if we represent the sheet displacement

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Consider **perturbations** of the form

$\varphi'_1, \varphi'_2 = F(z)\ e^{i(kx)+\sigma t}$ and $\zeta = A\ e^{i(kx)+\sigma t}$

These are Fourier components or normal modes! What is F(z) ?

Condition at infinity: the amplitude of the perturbations goes to zero!

Since $\Delta\varphi'_i=0$ $\varphi'_i = B_1 e^{-kz} + B_2 e^{kz}$

$\varphi'_i \to 0$ for z —› + ∞ thus for **z>0** **$B_2=0$**
$\varphi'_i \to 0$ for z —› – ∞ thus for **z<0** **$B_1=0$**

We can now **solve the form of ζ^*, φ^*_1, φ^*_2 with amplitudes A, B_1, and B_2**

$\zeta = \mathbf{A}\ e^{ikx+\sigma t}$,

$\varphi'_1 = B_1 e^{-kz} e^{ikx+\sigma t}$. $\varphi'_2 = B_2 e^{kz} e^{ikx+\sigma t}$

Substitution in conditions I and II:

$-kB_1 = (\sigma - \frac{1}{2} i k U) A$
$-kB_2 = (\sigma + \frac{1}{2} i k U) A$

and condition III: $i\,[\sigma(B_2 - B_1)_{z=0} + \frac{1}{2} U (B_2 k + B_1 k)_{z=0}]\ e^{i(kx)} = H$

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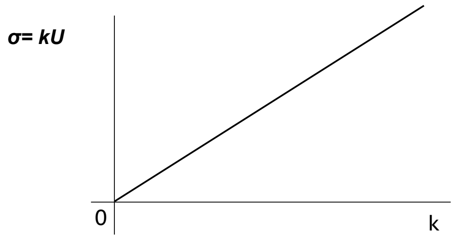
With ***Im(H)=0*** we obtain:

$\sigma = \frac{1}{2} ik(U_1+U_2) \pm \frac{1}{2} k(U_1 - U_2)$

for **$U_1 = -U_2$** this reduces to

$\sigma = \pm kU$

- exponential growth for any velocity for $\sigma>0$
- growth rate depends on U



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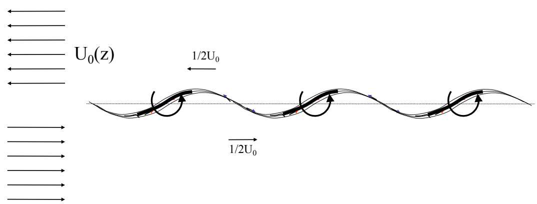
$\sigma = \pm kU$

$\sigma(k)$ is the dispersion relation showing the variation of growth rate with k. For $\sigma>0$, $k\neq0$ the sheet is unstable. Small wavelengths grow faster than short ones.

All wave lengths are unstable no matter how small U is!. In reality often there is a cutoff for small wavelengths as we will see later.



(Vertical velocity is out phase with pressure and velocity, and horizontal vorticity $\omega_y \approx -\frac{\partial w}{\partial x} = -ikw$)

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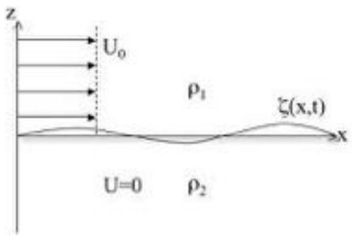
Laminar basic flow ; with layers 1 and 2 of different density



Viscous effets are considered negligible and the fluid is incompressible.
This flow satisfies the Euler equations, continuity and hydrostatic balance.

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Perturbations

The basic flow satisfies
The Euler equations, continuity and hydrostatic balance are ;

$$\frac{\partial \vec{u}}{\partial t} + u.\nabla\vec{u} = -\frac{\nabla p}{\rho} - g \qquad \nabla.\vec{u} = 0 \qquad \frac{dp}{dz} = -\rho g$$

We suppose a perturbation of the form

$$p = P + p'$$
$$\rho = \rho_i + \rho_i' \quad \text{(i=1,2)}$$
$$u_1 = U_0 + u_1'$$
$$u_2 = u_2'$$

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The basic flow is given by

$$(u_1, w_1) = (U_0, 0) \qquad (u_2, w_2) = (0, 0)$$
$$p(z) = P - \rho_1 g z \ \ (z > 0) \qquad p(z) = P - \rho_2 g z \ \ (z < 0)$$

Substitue the perturbations (neglect second order terms), so that we obtain :

$$\nabla.(\bar{U}_0 + \bar{u}') = 0$$
$$\frac{\partial \bar{U}_0 + \bar{u}'}{\partial t} + (\bar{U}_0 + \bar{u}')\frac{\partial(\bar{U}_0 + \bar{u}')}{\partial x} = \frac{\nabla(P + p')}{\rho_0 + \rho'}$$
$$\Longrightarrow$$

For the upper layer we obtain :   Lower layer:

$$\frac{\partial u_i'}{\partial x} + \frac{\partial w_i'}{\partial z} = 0 \quad (i = 1, 2) \qquad (1)$$
$$\frac{\partial u_1'}{\partial t} + U_0\frac{\partial u_1'}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1'}{\partial x} \quad \Big| \quad \frac{\partial u_2'}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial x} \qquad (2)$$
$$\frac{\partial w_1'}{\partial t} + U_0\frac{\partial w_1'}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1'}{\partial z} \quad \Big| \quad \frac{\partial w_2'}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial z} \qquad (3)$$

We use perturbations of the form

$$(u', w', p', \zeta') = (\hat{u}, \hat{w}, \hat{p}, \hat{\zeta})(z)e^{ikx - i\omega t}$$

The fonction $(\hat{u}, \hat{w}, \hat{p}, \hat{\zeta})(z)$ can be derived from eqs. (1,2 and 3). With $\frac{\partial(2)}{\partial x} + \frac{\partial(3)}{\partial z} = -\nabla^2 p_i'$ and continuity one obtains $\nabla^2 p_i' = 0$. Using the expression for the perturbations above yields

$$\frac{\partial^2 p_i'}{\partial z^2} - k^2 p_i' = 0,$$

with solutions $p_i' = A_i e^{kz} + B_i e^{-kz}$.
Under the condition that perturbations disappear with distance from the interface $z \to \pm\infty$ $\quad \hat{p}' \to 0$ we obtain

In layer 1 : $(u', w', p', \zeta')_1 \sim e^{-kz} e^{i(kx - \omega t)}$

In layer 2 : $(u', w', p', \zeta')_2 \sim e^{kz} e^{i(kx - \omega t)}$

Note : The basic equations provide information about the phase of the pressure with respect to the vertical motion. Substitution of the perturbations in the latter equation shows (omitting primes)

$$-i(\omega - kU_0)w_1 = -\frac{k}{\rho}p_1$$
$$-i\omega w_2 = -\frac{k}{\rho}p_2$$

(Vertical velocity is out phase with pressure and velocity, and horizontal vorticity $\omega_y \approx -\frac{\partial w}{\partial x} = -ikw$)

Interface conditions

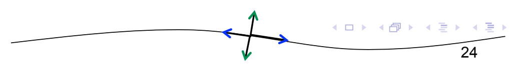
Lagrangian motion of a particle at the interface

I) Kinematic interface condition : particles remain at the interface. Consider a particle at the interface $\zeta(x, t)$, given by $z = \zeta(x, t)$. By continuity, the vertical motion of this particle should match the velocity above and below the interface :

$$\text{upper layer} \quad \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + U_0\frac{\partial \zeta}{\partial x} = w_1$$
$$\text{lower layer} \quad \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} = w_2$$

II) Dynamic condition : continuity of forces across the interface. Here, normal to the interface, pressure and gravity

$$p_1 - p_2 = (\rho_1 - \rho_2)g\zeta \quad \text{for } z = 0$$



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force balance **normal** to the interface

We consider the motion in the vertical direction :

$$\frac{\partial \zeta}{\partial t} + U_0 \frac{\partial \zeta}{\partial x} = w_1$$
$$\frac{\partial \zeta}{\partial t} = w_2$$
$$p_1 - p_2 = (\rho_1 - \rho_2) g \zeta$$
$$\frac{\partial w_2'}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial z}$$
$$\frac{\partial w_1'}{\partial t} + U_0 \frac{\partial w_1'}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1'}{\partial z}$$

Substitute the perturbations and write in matrix form to determine the dispersion relation.

$$i(kU_0 - \omega)\zeta - W_1 = 0$$
$$-i\omega\zeta - W_2 = 0$$
$$g(\rho_2 - \rho_1)\zeta + P_1 - P_2 = 0$$
$$-i\omega W_2 + \frac{k}{\rho_2} P_2 = 0$$
$$i(kU_0 - \omega) W_1 + \frac{k}{\rho_1} P_1 = 0$$

Elimination of W_1, W_2 and P_1, P_2 provides an equation in ζ

Solution

Sometimes it is easier to write this in the form of a matrix

$$\begin{pmatrix} i(kU_0-\omega) & -1 & 0 & 0 & 0 \\ -i\omega & 0 & -1 & 0 & 0 \\ g(\rho_2-\rho_1) & 0 & 0 & 1 & -1 \\ 0 & 0 & -i\omega & 0 & k/\rho_2 \\ 0 & i(kU_0-\omega) & 0 & k/\rho_1 & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ W_1 \\ W_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

If Det=0 then nontrivial solution exist. If there are many equations make use of a program like Python, Maple, Scylab, Matlab, or Mathematica. This provides the dispersion relation $\omega(k)$:

$$(\rho_1 + \rho_2)\omega^2 - 2kU_0\rho_1\omega + k^2 U_0^2 \rho_1 - kg(\rho_2 - \rho_1) = 0.$$

Interpretation 1

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2 U_0^2 \rho_1\rho_2 - kg(\rho_2 - \rho_1)(\rho_2 + \rho_1)}}{(\rho_1 + \rho_2)}$$

Remind that the form of the perturbation is $\sim e^{i(kx - \omega t)}$

- Water-Air interface : $U_0 = 0$ et $\rho_1 = 0$

  From the dispersion relation we obtain $\mathrm{Im}(\omega) = 0$, and $\mathrm{Re}(\omega)$:

  $$\omega = \pm\sqrt{kg}$$

  $\omega_i = 0 \rightarrow e^{\omega_i t} = 1 \rightarrow$ stable.

  $\omega_r \neq 0 \rightarrow$ surface waves with phase velocity : $c = \sqrt{g/k}$.

Interpretation 2

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1)}}{(\rho_1+\rho_2)}$$

- Stable fluid interface, but without shear
  i.e. $U_0 = 0$ and $\rho_1 > 0$

  The dispersion relation reduces to (only Re(ω)$\neq$ 0) :

$$\omega = \pm\sqrt{\frac{kg(\rho_2-\rho_1)}{(\rho_1+\rho_2)}}$$

$\rho_1 < \rho_2 \rightarrow \omega_i = 0$ stable ($\rho_1 > \rho_2$ instable)
The phase velocity is for interfacial gravity waves :

$$c = \pm\sqrt{\frac{g}{k}\frac{(\rho_2-\rho_1)}{(\rho_1+\rho_2)}} \qquad =\pm\sqrt{g'/k}$$

Interpretation 3

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1)}}{(\rho_1+\rho_2)}$$

- Stable density interface with shear $\rho_1 \neq \rho_2$, $U_0 \neq 0$, $\rho_2 > \rho_1$

  There is stability when :

$$U_0^2 \leq \frac{g}{|k|\rho_1\rho_2}(\rho_2^2-\rho_1^2)$$

There is instability when $\pm\omega_i \neq 0$, i.e. for

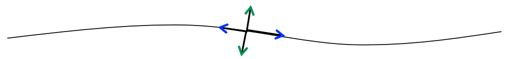
$$4k^2U_0^2\frac{\rho_1\rho_2}{\bar{\rho}^2} - 2kg\frac{\Delta\rho}{\bar{\rho}} \approx 2k(2kU_0^2 - g\frac{\Delta\rho}{\bar{\rho}}) > 0$$

From this expression, derive instability for (a typical length scale L).

$$Ri = \frac{-g}{\rho_0}\frac{\frac{d\rho}{dz}}{(\frac{du}{dz})^2} < \frac{1}{4}$$

EXERCISE

Modify the dispersion relation for a surface tension T.
(note that we only consider the force perpendicular
to the interface and not the forces tangential to the interface)



La condition de pression linéarisé à l'interface donne

$$p_2 - p_1 = (\rho_2-\rho_1)g\zeta - T\frac{d^2\zeta}{dz^2} \quad \text{pour } z = 0$$

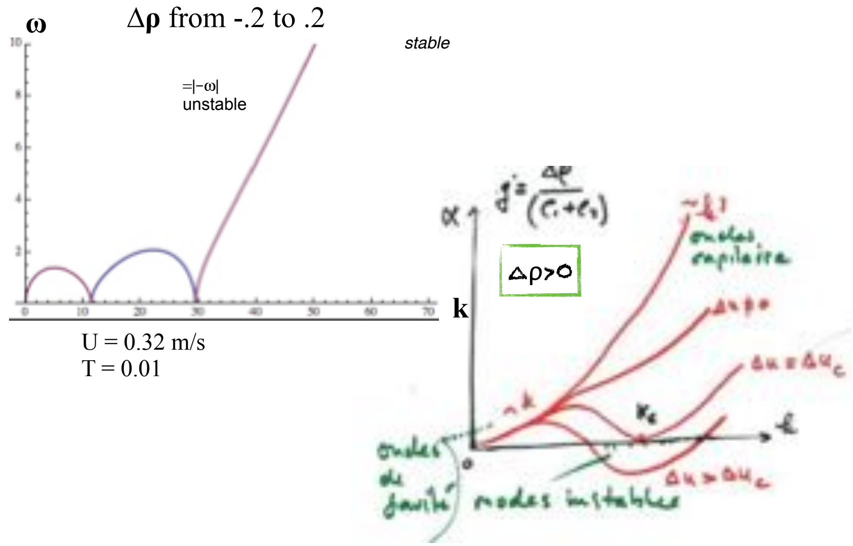
$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1) - k^3T}}{(\rho_1+\rho_2)}$$

$e^{-i\omega t}$

growth $Re(-i\omega t) > 0$

waves $Im(-i\omega t) \neq 0$

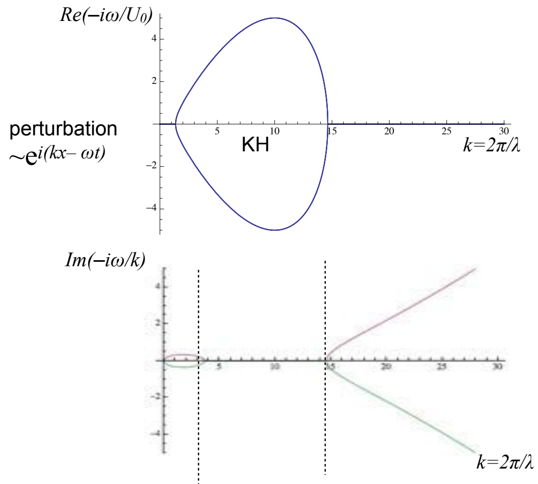
$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1) - k^3T}}{(\rho_1+\rho_2)}$$



Graphical representation

Δρ>0, stable stratification

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_2+\rho_1) - k^3T}}{(\rho_1+\rho_2)}$$



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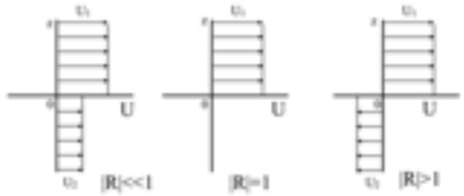
comments

Effect of velocity ratio $R = \frac{\Delta U}{2\bar{U}}$ on vortex sheet profile



R=ΔU/2U

convection speed $(U_1 + U_2)/2$

growth $\Delta U = (U_1 - U_2)$

The velocity ratio R is important to the nature of the instability :

For KH flow the interface conditions (pressure / continuity w) impose the dispersion relation :

$$(c - U_1)^2 + (c - U_2)^2 = 0$$

with solution $c \equiv \frac{\omega}{k} = \bar{U} \pm i\frac{\Delta U}{2} = \bar{U}(1 \pm iR)$ and $c_r = \bar{U}$ the propagation speed of the waves.

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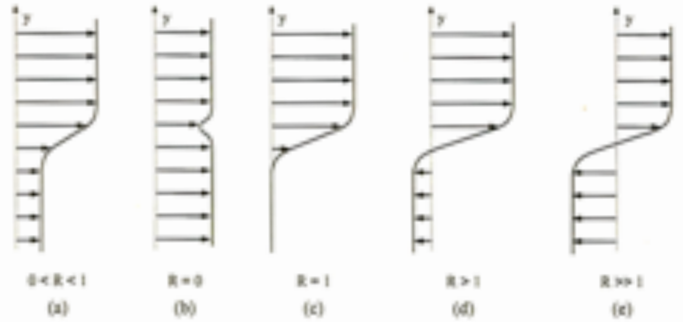
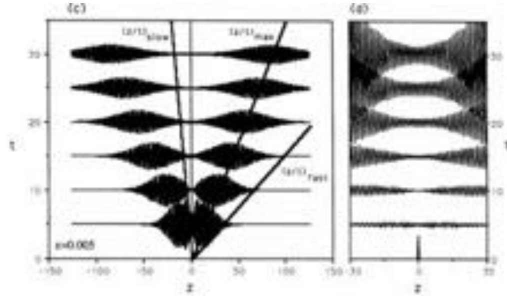


Fig. 2.6. Typical mixing layer velocity profiles as a function of velocity ratio R when $U_1 \geq |U_2|$: (a) $0 < R < 1$; (b) $R = 0$; (c) $R = 1$; (d) $R > 1$; (e) $R \gg 1$.

R = 1 : lower stream at rest; increase in layer thickness proportional to R
R<<1 : convection velocity >> growth ΔU
R > 1 : Counter flow mixing layers
R>>1 : well defined KH billows at one position
NOISE: mixing layer is a noise amplifier.

See Godreche & Manneville 1998

$|R| << 1$ weak shear ; simple linear relation between spatial and temporal development of instability
$|R| \geq 1$ complex relation between spatial and temporal development of instability.



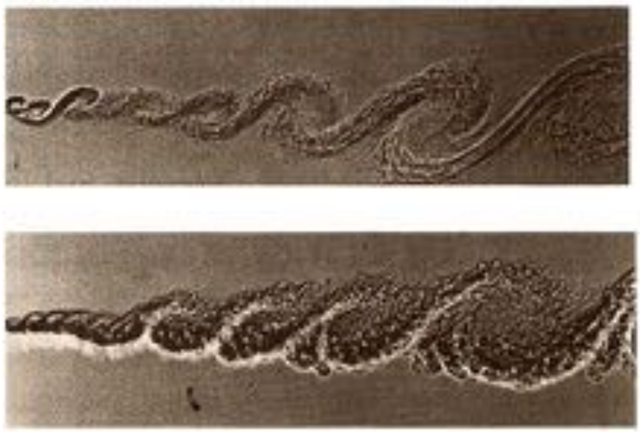
Convective versus absolute instability

The mixing layer

shear layer thickness $\delta(x) = \frac{(U_1+U_2)}{(dU/dy)_{max}}$

δ increases with x by diffusion ; vortex roll-up and vortex merging.

δ becomes linear in x far downstream.



Control numbers are : $Re = \frac{(U_1-U_2)\delta(0)}{\nu}$ and R = ΔU/2U

R=0 : no net shear (e.g. wake behind a flat plate)
δ increases proportionally with shear intensity R (growth rate)
R«1 slow streamwise development
$R \approx 1$ or $R \geq 1$, roll-up and merging occur closer to x=0.

Strouhal number describing the characteristic flow oscillation, the frequency of vortices of wave length

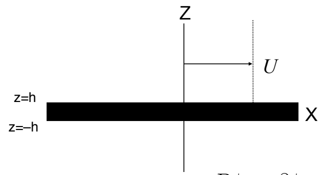
$$St_n = \frac{f_n \delta(0)}{\bar{U}} \approx 0.03$$

f_n is the natural vortex frequency in the wake

Vorticity layer instability

Viscosity: diffusive effects !

Consider the instability of the vorticity layer at the interface (2D)



vorticity ζ layer at z=0

Vorticity from Euler equations: $\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + U.\nabla\zeta = 0 \qquad \zeta = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$

Continuity $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

Basic flow is *U* in *x*-direction, i.e. (*U,0*)
Perturbation (*u',w'*) —› *(u,w)=(U+u', w')*

$$\zeta = \frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} - \frac{\partial U}{\partial z}$$

$$\frac{\partial \zeta}{\partial t} + (U+u')\frac{\partial \zeta}{\partial x} + w'\frac{\partial \zeta'}{\partial z} = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

Consider the instability of the vorticity layer at the interface (2D)

Linearise, neglect terms of second order

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{\partial^2 U}{\partial z^2}w' = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

Consider a perturbation of the form $u', w' \longrightarrow (u'(z), w'(z))\, e^{ikx + i\sigma t}$,

$$i(\sigma + kU)\left(ikw' - \frac{\partial u'}{\partial z}\right) - \frac{d^2U}{dz^2}w' = 0$$

$$iku' + \frac{\partial w'}{\partial z} = 0$$

Eliminate *u'* to find *THE* ordinary differential equation in *z* to solve:

$$(\sigma + kU)\left(\frac{\partial^2 w'}{\partial z^2} - k^2 w'\right) - \frac{d^2U}{dz^2}kw' = 0$$

Since for z=0, (the region of interest), *dU/dz* is discontinuous we have to replace this differential with the difference Δ across the two layers:

$$\lim_{\Delta\to 0}\int_{-\Delta/2}^{\Delta/2}(\sigma + kU)\left(\frac{\partial^2 w'}{\partial z^2} - k^2 w'\right) - \frac{d^2U}{dz^2}kw' \;\; dz =$$

$$(\sigma + kU)\Delta\frac{\partial w'}{\partial z} - \Delta\frac{\partial U}{\partial z}kw' = 0$$

Note: we have used here w' continuous across the interface

Move with the fluid, i.e. **u** = U/2 for z > h
and **u** = – U/2 for z < –h

For z>h and z<–h $d^2U/dz^2=0$, we have (as before)

$$\frac{\partial^2 w'}{\partial z^2} - k^2 w' = 0$$

So that for we obtain for the different layers (as before):

$$w' = Ae^{-kz} \qquad \text{for z>h}$$
$$w' = Be^{-kz} + Ce^{kz} \qquad \text{for -h<z<h}$$
$$w' = De^{kz} \qquad \text{for z<-h}$$

Continuity of w' at z>h and z<–h gives then

$$Ae^{-kh} = Be^{-kh} + Ce^{kh}$$
$$De^{-kh} = Be^{kh} + Ce^{-kh}$$

and the relation $(\sigma + kU)\Delta\frac{\partial w'}{\partial z} - \Delta\frac{\partial U}{\partial z}kw' = 0$ gives with $\boldsymbol{u} = \pm U/2$

$$2(\sigma + k\mathbf{u})Ce^{kh} - \frac{\mathbf{u}}{h}(Be^{-kh} + Ce^{kh}) = 0$$
$$2(\sigma - k\mathbf{u})Be^{kh} + \frac{\mathbf{u}}{h}(Be^{kh} + Ce^{-kh}) = 0$$

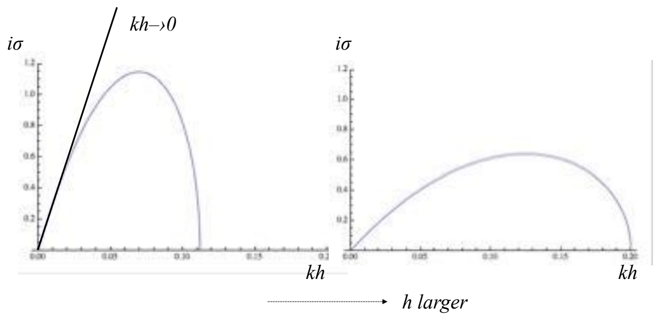
eliminate B and C gives then …

$$\sigma^2 = \frac{\mathbf{u}^2}{4h^2}\left[(2kh-1)^2 - e^{-4kh}\right]$$

in the limit of $kh \to 0$ $\sigma^2 = -k^2\mathbf{u}^2$ with $u', w' \sim e^{ikx + i\sigma t}$, we note that $i\sigma > 0 \longrightarrow$ growth !
Same as the KH interface from above.

For large values of kh shear layer thickness decreases the growth σ
$\sigma^2 = +k^2\mathbf{u}^2$ so that $\sigma = \pm k\mathbf{u}$; Since Im(σ)=0, stability

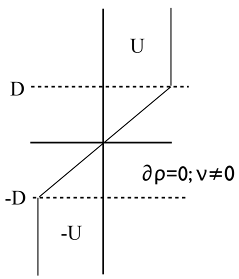
$$\sigma^2 = \frac{\mathbf{u}^2}{4h^2}\left[(2kh-1)^2 - e^{-4kh}\right]$$



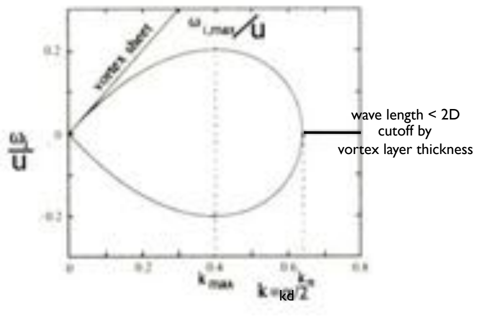
cut off wave number

(same result as above)

Effect of viscosity on the instability of a shear layer

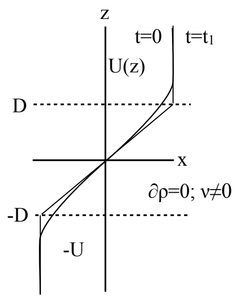


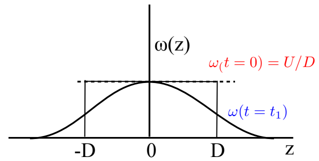
for Re=Ud/ν –> ∞
instability for a band of wavelengths 0<2kd<1.3
maximum growth at 2kd=0.8



What is the effect of viscosity on :
- growth rate ?
- wavenumber ?

viscous spreading of the shear layer





$\omega_(t=0) = U/D$

$\omega(t=t_1)$

$$\left.\begin{array}{r}\omega = \nabla \times U \\ U = U(z)\end{array}\right\} \to \omega = (0, \frac{\partial U(z)}{\partial z}, 0)$$

$$\frac{\partial \bar{\omega}}{\partial t} = \nabla \times (\bar{u} \times \bar{\omega}) + \nu\nabla^2\bar{\omega} \qquad \nabla \times (\bar{u} \times \bar{\omega}) = \bar{u}(\nabla.\bar{\omega}) - \bar{\omega}(\nabla.\bar{u}) + (\bar{\omega}.\nabla)\bar{u} - (\bar{u}.\nabla)\bar{\omega}$$

$$\frac{\partial \bar{\omega}}{\partial t} = -\bar{u}.\nabla\bar{\omega} + \bar{\omega}.\nabla\bar{u} + \nu\nabla^2\bar{\omega}$$

$\bar{u} \perp \bar{\omega} \to \bar{\omega}.\nabla\bar{u} = 0$

uni directional flow U(z) $\to \bar{u}.\nabla\bar{\omega} = 0$

$$\to \frac{\partial \omega}{\partial t} = \nu\nabla^2\omega \qquad \omega = \omega_y$$

vorticity diffusion equation

The viscous spreading of the shear layer

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \text{ here } \omega = \omega_y$$

For a *thin shear layer* of thickness δ(y) and amplitude U

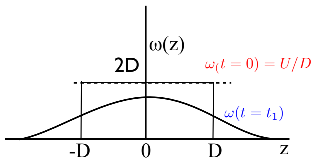
$$\omega(y, t=0) = U\delta(y)$$

the *solution* of the diffusion equation is (see Batchelor 1969)

$$\omega(y,t) = \frac{U}{2\sqrt{\pi \nu t}} e^{-y^2/4\nu t}$$

for a *shear layer from y = −d to +d* $\omega(y,t) = \int_{-d}^{+d} \frac{\omega(y-y',t)}{2d} dy$

the solution is $\omega(y,t) = \frac{U}{2d}[erf\left(\frac{y+d}{\sqrt{4\nu t}}\right) - erf\left(\frac{y-d}{\sqrt{4\nu t}}\right)]$ with $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$



$$\omega(y, t \to 0) = 0 \text{ but } \int_{-\infty}^{+\infty} \omega(y,t) dy = 2U$$

velocity jump across the layer is maintained for all times; vorticity decays by viscous damping.

Determine the speed of the spreading =>

Thickness of the diffusing shear layer.

The standard deviation of the vorticity distribution at t=0 is

$$\sigma^2 = \frac{1}{2u} \int_{-\infty}^{+\infty} y^2 \omega(y,t) dy$$

this is generally smaller than the real distribution (here 2D) so rescale:

$$\Delta^2 = \frac{a}{2u} \int_{-\infty}^{+\infty} y^2 \omega(y,t) dy$$

With $a = \left(\frac{D}{\sigma}\right)^2$ so that at *t=0* $\Delta^2 = D^2$

For a linear velocity profile a=3 (at t=0). The integral then yields

$$\Delta^2 = D^2 + \delta^2 \qquad \delta = \frac{3}{2}\sqrt{4\nu t}$$

The spreading of the vorticity distribution can be written then as

$$\boxed{\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{2\delta}{D^2+\delta^2}\frac{d\delta}{dt}}$$

two cases. 1) weak viscous spreading δ/D<<1 an
2) thin layer with strong viscous effects, i.e. δ/D>>1

If δ/D<<1 viscous effects are small at t=0, initial thickness is large

$$\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{2\delta}{D^2+\delta^2}\frac{d\delta}{dt} \approx \frac{2\delta}{D^2}\frac{d\delta}{dt} \approx \frac{2\nu}{D^2} \quad \text{=constant in time}$$

$$\delta \sim \sqrt{\nu t}$$

If δ/D>>1 t=0, thin layer with strong viscous effects (Δ≈δ)

$$\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{2\delta}{D^2+\delta^2}\frac{d\delta}{dt} \approx \frac{1}{\delta}\frac{d\delta}{dt} = \frac{1}{t}$$

Now compare with the growth rate of the instability *(Re= Real part)*

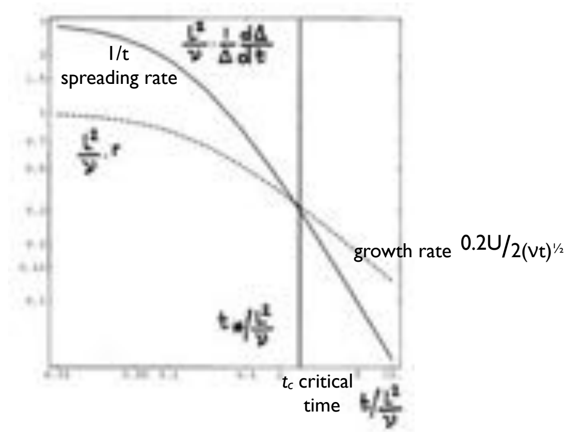
$Re(-i\sigma) = \frac{0.2U}{\Delta}$ which is the maximum growth rate for the inviscid case

this growth rate is affected by viscosity due to increase in thickness Δ,
in case δ/D<<1 Δ=D and the growth rate, 0.2 U/D, is not affected.

In case Δ≈δ the spreading of the viscous layer is faster than the growth of the instability.

$Re(-i\sigma) = \frac{0.2U}{D} \approx \frac{0.2U}{\delta} = \frac{0.2U}{2\sqrt{\nu t}}$ and spreading of layer is $\frac{1}{\delta}\frac{d\delta}{dt} = \frac{1}{t}$

spreading rate of the vorticity



At **critical time** $t_c = 100\, \nu/U^2$ –> $\delta_c = (\nu t_c)^{1/2} = 10\nu/U$

==> Critical time depend on Reynolds number UD/ν

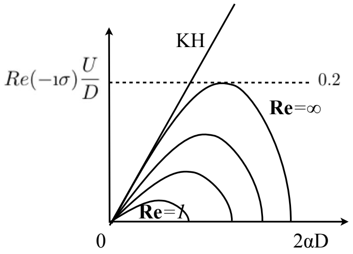
Critical time t_c as a function of Reynolds number **Re**

with $\delta_c = (\nu t_c)^{1/2} = 10\nu/U$ one can write $\mathbf{Re} = UD/\nu = D/(\delta_c/10)$ so that $\delta/D = 10/\mathrm{Re}$

For the growth rate we obtain: $Re(-\imath\sigma) = \frac{0.2U}{\sqrt{D^2+\delta^2}} = \frac{U}{D}\frac{0.2}{\sqrt{1+\delta^2/D^2}}$

so that the non-dimensional growth is

$$Re(-\imath\sigma)\frac{U}{D} = \frac{0.2}{\sqrt{1+(10/\mathbf{Re})^2}}$$



Villermaux 1998 Phys. of Fluids
Betchov Szewczyk Phys. of Fluids 1963

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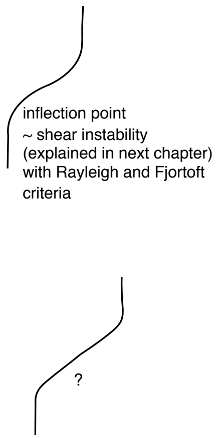

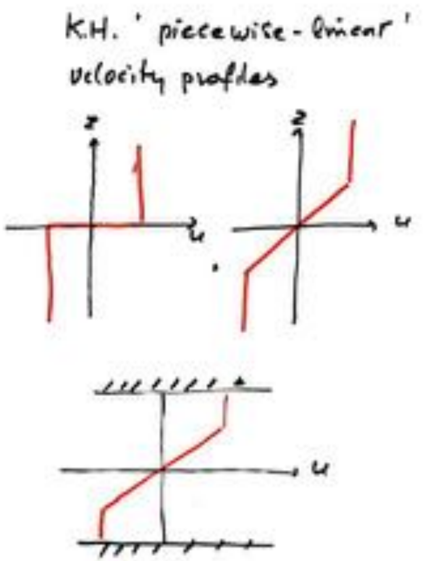

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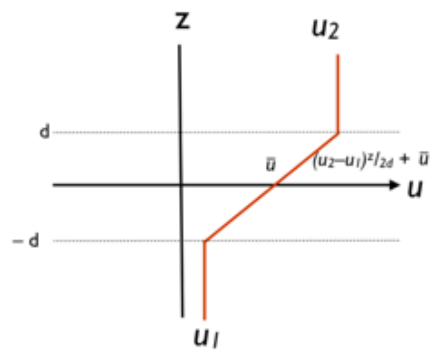
Shear layer with non-zero thickness (Rayleigh 1869)



We take for simplicity $\bar{x} \to (x, z)$ (ρ=constant)
pressure $p = p(z)$
Basic flow + perturbations : $(U_0 + u, w, P_0 + \delta p)$

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- write down adapted (2D) Euler equations and basic state
- derive perturbation equations
- define the form of the perturbation
- Substitute and obtain a PDE for w at z=0

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substitute in the Euler equations :

$$\frac{\partial u}{\partial t} + U_0\frac{\partial u}{\partial x} + w\frac{\partial U_0}{\partial z} = -\frac{1}{\rho}\frac{\partial \delta p}{\partial x}$$

$$\frac{\partial w}{\partial t} + U_0\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial \cancel{P_0} + \delta p}{\partial z} - \cancel{g}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Substitute perturbations : $v(x,z,t) = \hat{v}(z)exp\{i(kx+\omega t)\}$

$$i(\omega + kU_0)u + w\frac{\partial U_0}{\partial z} = -\frac{ik}{\rho}\delta p$$

$$i(\omega + kU_0)w = -\frac{1}{\rho}\frac{\partial}{\partial z}\delta p$$

$$u = \frac{i}{k}\frac{\partial w}{\partial z}$$

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boundary conditions

Reduce variables to obtain a partial differential equation in z (eliminate u with i and iii)

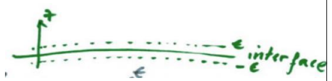
$$-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = -\frac{ik^2}{\rho}\delta p \qquad (4)$$

eliminate δp to obtain a single equation in w

$$\frac{\partial}{\partial z}\left[-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z}\right] = k^2(\omega + kU_0)w \qquad (5)$$

The *kinematic boundary condition* imposes that w is continuous across the interface :

$$\lim_{\epsilon \to 0}\int_{-\epsilon}^{\epsilon} w\, dz = 0$$



Applying this to equation (5) yields :

$$-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = 0 \qquad (6)$$

$\partial 4/\partial z$ is equal to the pressure gradient ; (6) implies $\delta p_1 - \delta p_2 = 0$ so that also the *dynamic boundary condition* is satisfied.

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Show that in regions where $\frac{\partial U_0}{\partial z} = 0$ we have $\frac{\partial^2 w}{\partial z^2} - k^2 w = 0$ ...
In the three regions we have :

$z > d$	$w = A_+ e^{-kz}$
$-d < z < d$	$w = A_0 e^{-kz} + B_0 e^{kz}$
$z < -d$	$w = A_- e^{kz}$

with the constants A_-, A_+, A_0 and B_0 to determine with the continuity accros the interface, i.e.
1) Kinematic boundary condition : continuity of w at $\pm d$
2) Continuity of pressure gives :

$$-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = 0$$

(suppose $U_2 = U$ and $U_1 = -U$ and the relation found with *1)*)

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Continuity of w at $\pm d$:

$$+d : A_+ e^{-kd} = A_0 e^{-kd} + B_0 e^{+kd}$$
$$-d : A_- e^{-kd} = A_0 e^{kd} + B_0 e^{-kd}$$

gives with continuity of $-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = 0$:

$$+d : 2(\omega + kU)B_0 e^{kd} - \frac{U}{d}(A_0 e^{-kd} + B_0 e^{kd}) = 0$$
$$-d : 2(\omega - kU)A_0 e^{kd} + \frac{U}{d}(A_0 e^{kd} + B_0 e^{-kd}) = 0$$

Elimination of $\frac{A_0}{B_0}$ yields the dispersion relation for ω
(Rayleigh 1896 vol11, p 393 and Drazin p 146 :

$$\omega^2 = \frac{U^2}{4d^2}\left[(1 - 2kd)^2 - e^{-4kd}\right]$$

since $\sim \exp[i(kx + \omega t)]$ instability for $i\omega > 0$.

Simplify the dispersion relation $\alpha = 2kd$ and $\Omega = \omega/(2kU)$
Since $U_1 = -U_2 = -U$, the phase velocity is $c = \omega/k$ (in case there is a mean velocity, it increases the phase velocity)

$$4\alpha^2\Omega^2 = (1-\alpha)^2 - e^{-2\alpha}$$

so that :

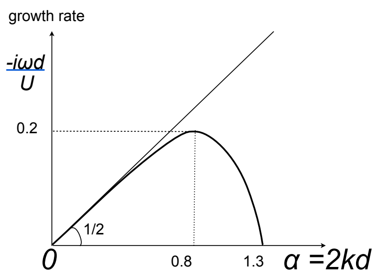
$$\Omega^2 = 1/4\frac{\left[(1-\alpha)^2 - e^{-2\alpha}\right]}{\alpha^2}$$

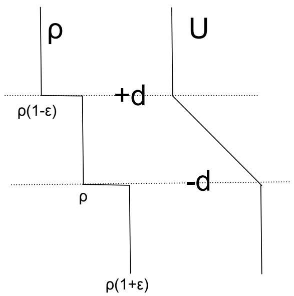
deduce Kelvin Helmholtz instability, i.e. $d \to 0$,

$$\omega = ikU$$

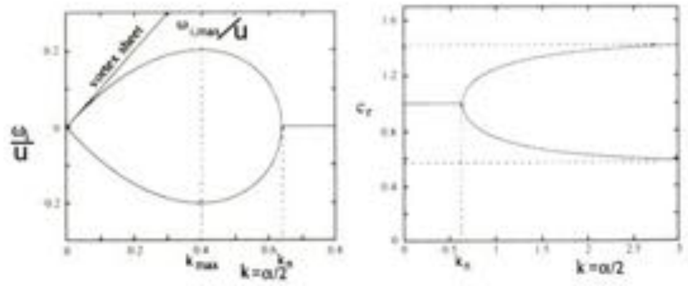
$$\omega^2 = \frac{U^2}{4d^2}\left[(1 - 2kd)^2 - e^{-4kd}\right]$$

Kelvin Helmholtz





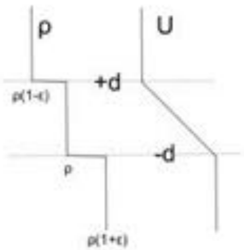
$\Omega^2 = \frac{1}{4\alpha^2}\left[(1-\alpha)^2 - e^{-2\alpha}\right]$ and $\Omega = \omega/(2kU)$ and $c_r = \omega_r/k$



Large wave lengths (small k) do not see the thickness of the interface and are unstable as KH
Short wave lengths are stabilized (large k), they are within the shear layer.

With density distributions I

(see Chandrasekhar 1961, p488-489)



$$\begin{array}{ll} z > d & \rho = \rho_0(1-\epsilon) \\ -d < z < d & \rho = \rho_0 \\ z < -d & \rho = \rho_0(1+\epsilon) \end{array}$$

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the dispersion relation reads :

$$e^{-2\alpha} = \left[1 - \frac{\alpha(\Omega+1)^2}{J+(\Omega+1)+\epsilon\alpha/2(\Omega+1)^2}\right]\left[1 - \frac{\alpha(\Omega-1)^2}{J-(\Omega-1)-\epsilon\alpha/2(\Omega-1)^2}\right]$$

with $\Omega = \omega/(kU)$ and J the Richardson number :

$$J = \frac{\epsilon g k}{2U^2 k} \sim \frac{g\Delta\rho/2d}{\rho(dU/dz)^2}$$

For stability $Re(\Omega^2) > 0$. Unstable when $Re(\Omega^2) < 0$ i.e. when

$$\frac{k}{1+e^{-k}} < J+1 < \frac{k}{1-e^{-k}}$$

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The Richardson (J) number represents the ratio between the kinetic energy of relative motion $(\frac{\partial U}{\partial z})^2$ and the work that must be done to overcome the restoring buoyancy force.

(see exercise on particle displacement of lecture 1 ; note that in this exercise the Boussinesq approximation is used by assuming that $\Delta\rho U$ is small). The results for instability is :

$$Ri(=J) = \frac{-g}{\bar{\rho}}\frac{d\rho/dz}{(dU/dz)^2} = \frac{\text{buoyancy force}}{\text{inertia force}} < \frac{1}{4}$$

Exercise :

Consider a basic flow with velocity profile $U(z)$ and density distribution $\rho(z)$ and neglect viscous effects. Derive the dispersion relation (Taylor-Goldstein equation).

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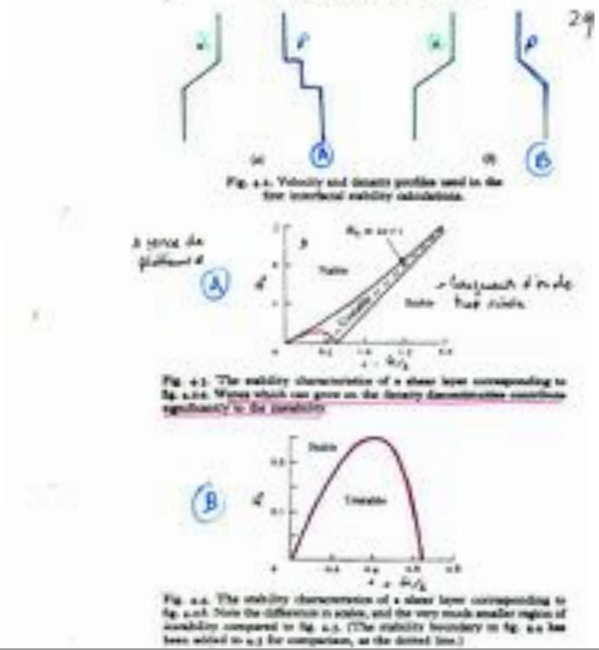

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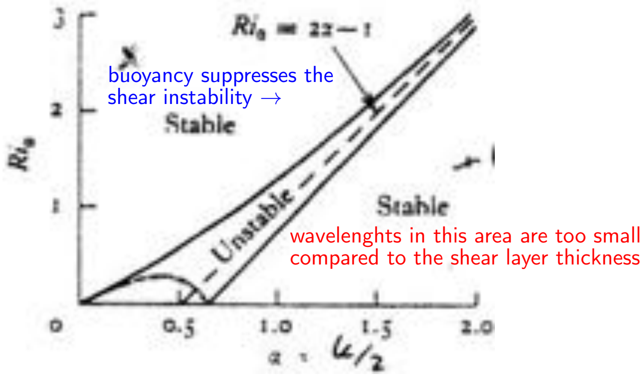

instability for $\frac{k}{1+e^{-k}} < J + 1 < \frac{k}{1-e^{-k}}$

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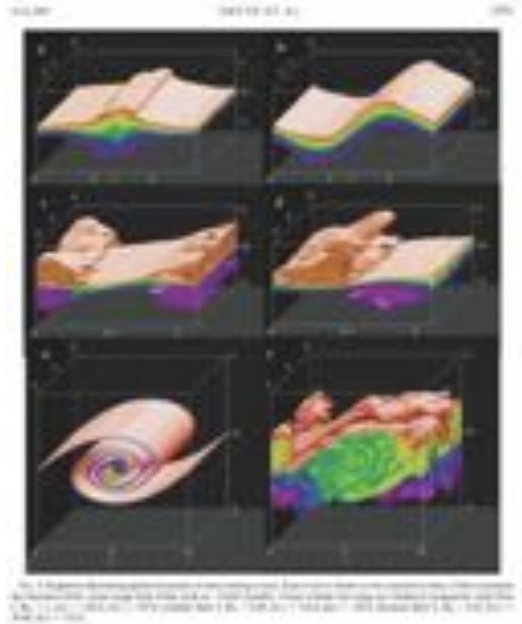

Ri =1

Ri =0.45

Ri =0.15

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Hölmböe instability

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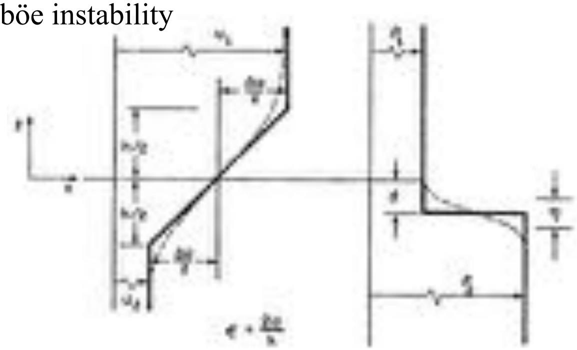
Hölmböe Instability

Hölmböe instability



We take for simplicity again $\bar{x} \to (x, z)$
Density interface at the level $z = -d$; thickness η and shear layer with total thickness h (Lawrence et al Phys. of Fluids 1991).

We first consider waves $\overline{U} = 0$ and then instabilities for $\epsilon = 0$.

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Hölmböe instability : the mechanism
consider Rayleigh's 1896 shear flow for $U_{z=0} = 0$

Mutually interacting waves:

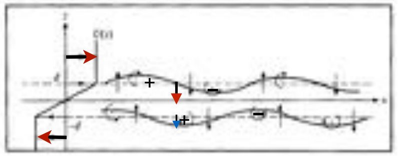
$$w_1 = i\omega\zeta_1 + ikU \rightarrow \frac{w_1}{kU} = i\frac{c}{U}\zeta_1 + i$$

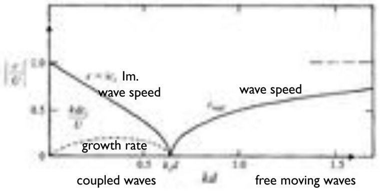
$$w_2 = i\omega\zeta_2 - ikU \rightarrow \frac{w_2}{kU} = i\frac{c}{U}\zeta_2 - i$$

$$\left(\frac{\omega/k}{U}\right)^2 = \left(\frac{c}{U}\right)^2 = \frac{(1-2kd)^2 - e^{-4kd}}{(2kd)^2}$$

instability when phase shift is λ/4
damping for phase shift λ/2

The shift is due to the Doppler effect on the oscillatory motion (c_r) -> resonance!





two types of instability:
1) stationary (i.e. non-oscillatory), for p'~ *exp*{*i*k(x-ct)} imaginary part =0 (c_r=0)

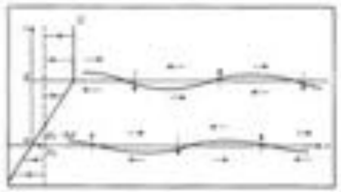
2) travelling wave on the vorticity interface and a standing wave on the density interface.

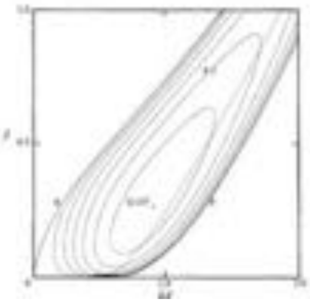
$$(c^2 - c_1^2)(c - (U_0 - c_2)) + c_1^2 c_2 e^{-kd} = 0$$

$$c_1^2 = g'/2k \text{ and } c_2 = U_0/2kd$$

same mechanism

The phase shift is due to the Doppler effect on the oscillatory motion

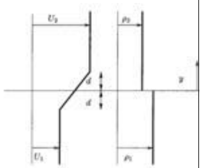




Holmboe (1962), Baines and Mitsudera (1994)

Suppose step-profile, symmetric interface ($\epsilon = 0$) in

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{JN^2}{(U-c)^2} - \frac{U''}{(U-c)}\right\}\phi = 0.$$



Then the dispersion relation reads

$$D(k,\omega,J,a) = (\omega - ak)^4 + n_2 k^2(\omega - ak) + n_0 k^4 = 0$$

where $a = U_{mean}/\Delta U$

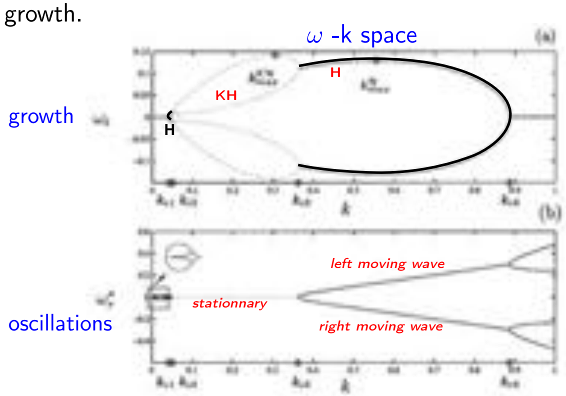
$$n_2 = \frac{-J}{sk} + \frac{e^{-4sk} - (2sk-1)^2}{4k^2} \text{ and } n_0 = \frac{J}{sk} + \frac{(e^{-2sk} + 2sk - 1)^2}{4k^2}$$

with $s = sgn(k_r)$. The roots are then (Ortiz et al POF 2002) :

$$\omega(k) = ak \pm \left\{\frac{-n^2k^2 \pm \Delta^{1/2}}{2}\right\}^{1/2} \text{ and } \Delta = (n_2k^2)^2 - 4n_0k^4$$

$$\omega(k) = ak \pm \left\{\frac{-n^2k^2 \pm \Delta^{1/2}}{2}\right\}^{1/2} \text{ and } \Delta = (n_2k^2)^2 - 4n_0k^4$$

advection with speed $a = U_m/\Delta Uk$ results in Doppler shift . To move with the local mean flow we should take $\omega_r^* = \omega_r - ak$ where $\omega = \omega_r + i\omega_i$; ω_r representing the oscillatory part and ω_i the growth.



four neutral waves:
2 propagate to the right (ω>0)2 to the left (ω<0)

Ortiz et al, Phys of Fluids 2002

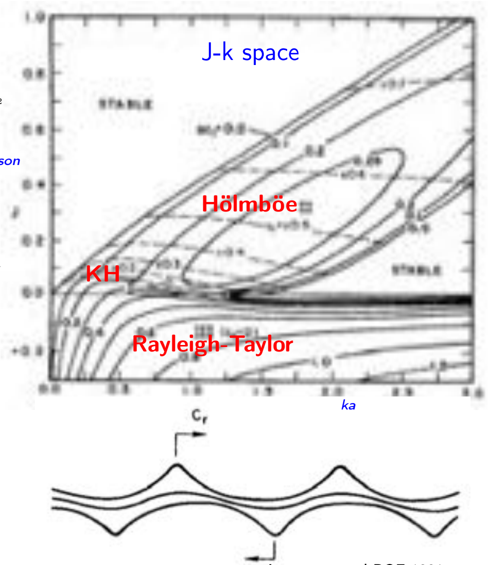
$$Ri \approx \frac{N^2 \Delta z^2}{\Delta U^2}$$

$$Ri = \frac{g \frac{\Delta \rho}{\bar{\rho} \Delta z_\rho}}{(\Delta U)^2} \left(\frac{\Delta z_u}{\Delta z_\rho} \right)^2$$

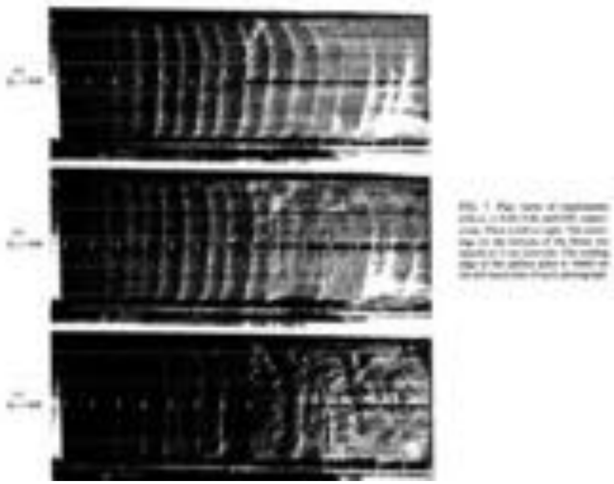
Richardson number

$$Ri \approx \frac{N^2 \Delta z^2}{\Delta U^2}$$

$$Ri = \frac{g \frac{\Delta \rho}{\bar{\rho} \Delta z_\rho} \Delta z_u^2}{(\Delta U)^2}$$



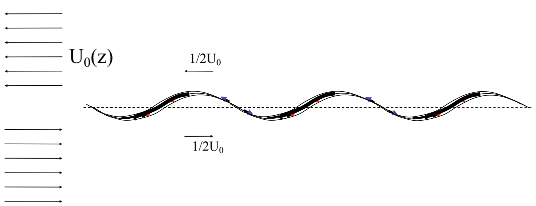
Lawrence et al POF 1991.



Lawrence, Browand & Redekop 1991

see movies ...

KH instability: vorticity layer rolls up at the interface



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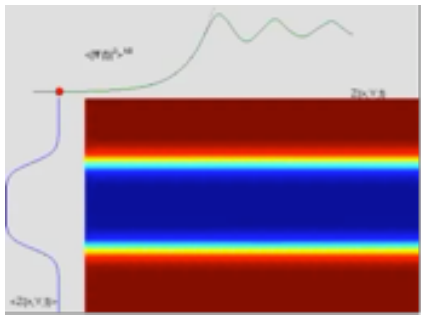
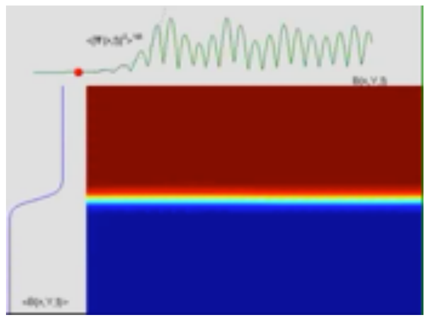
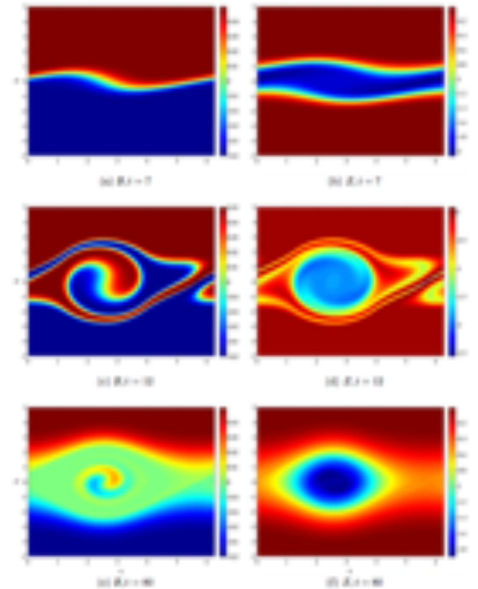




Neil Balmforth 2010, Woodshole



Roy Anubhab 2010, Woodshole

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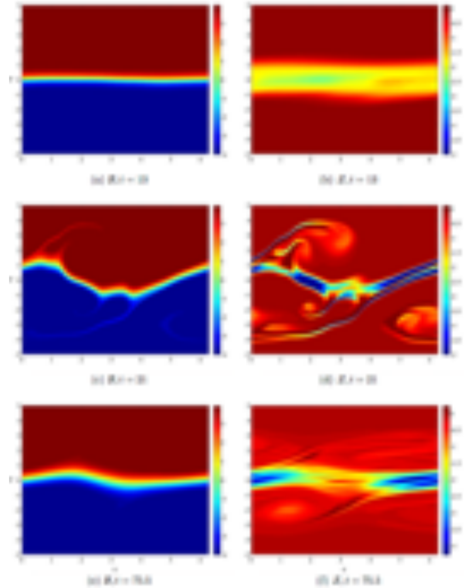

Roy Anubhab 2010, Woodshole

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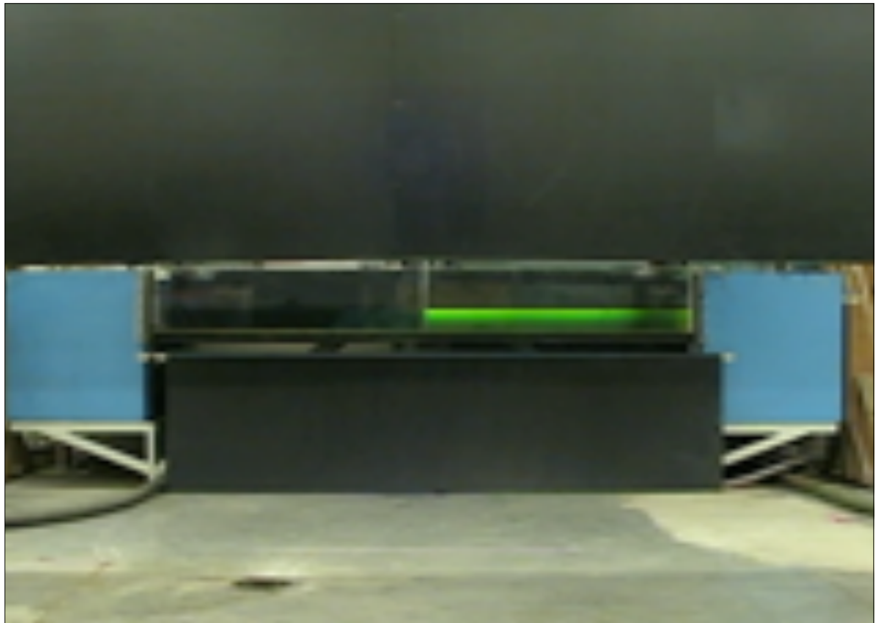

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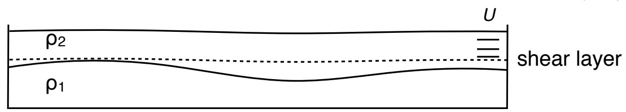
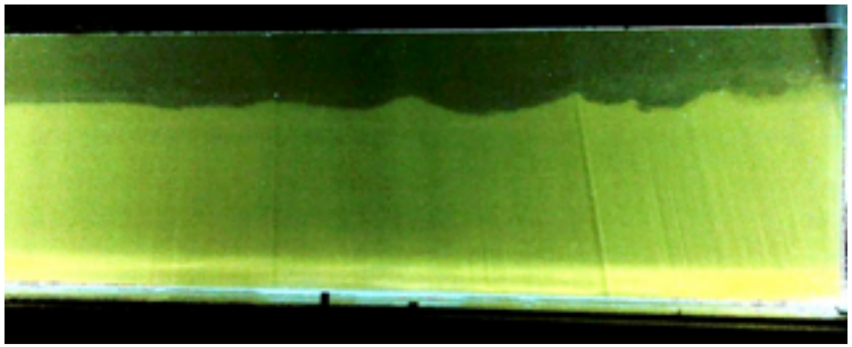
Measurements on KH instability

$$Ri \approx \frac{N^2 \Delta z^2}{\Delta U^2} \approx \frac{1}{4}$$

$$Ri = J\,R = \frac{g'\,\Delta z_u}{\Delta U^2}\left(\frac{\Delta z_u}{\Delta z_\rho}\right)$$





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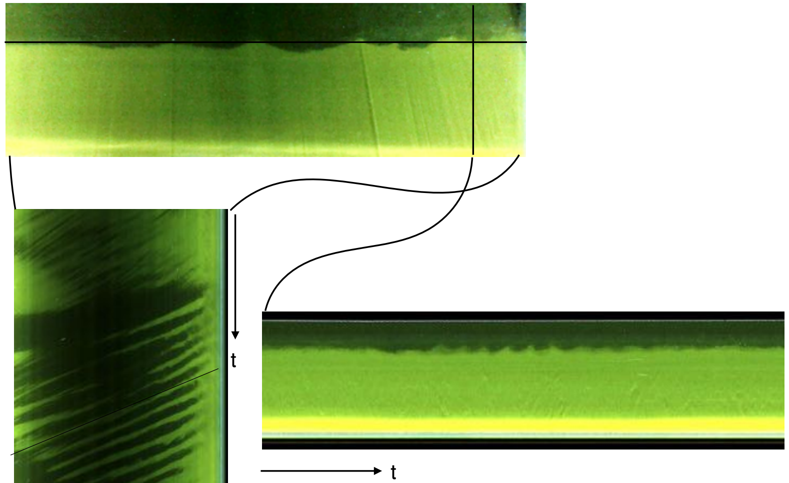
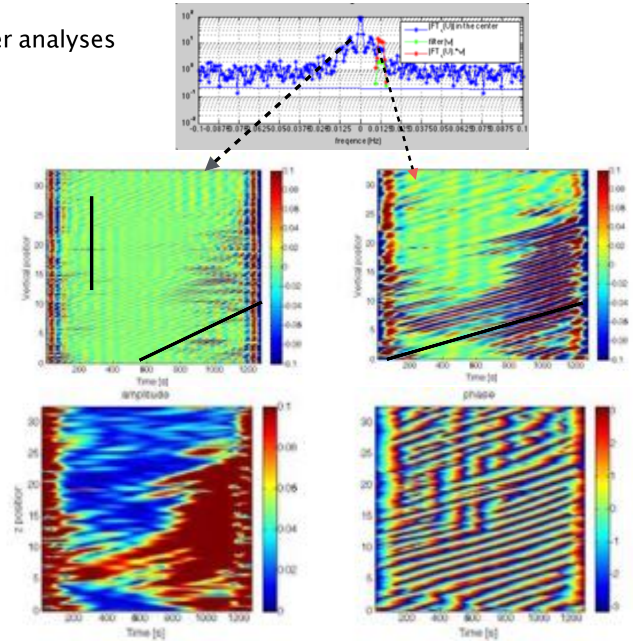
Space time diagram (also Hövmöller diagram)



With Fourier analyses, frequency , amplitude
Mercier et al. Phys of Fluids Vol 20, 2008.

Fourier analyses



Continuous velocity profiles.

Exercise :
Consider a basic flow with velocity profile $U(z)$ and density distribution $\rho(z)$ and neglect viscous effects. Derive the dispersion relation (Taylor-Goldstein equation).

The Taylor-Goldstein equation

- Parallel flow U(z) $[U+u', v', w']$ and stratification N.
- Euler Equations, viscosity $\nu = 0$
- Squires theorem $(\nu = 0)$: $3D \to 2D$, stronger growth for 2D than 3D (see later)
- linearize, define a stream function $u' = \frac{\partial \psi}{\partial z} \quad w' = -\frac{\partial \psi}{\partial x}$
- perturbation $[\rho, p, \psi] = [\hat{\rho}(z), \hat{p}(z), \hat{\phi}(z)]e^{[ik(x-ct)]}$

$$\rightarrow$$

$$(U-c)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{N^2}{(U-c)} - U_{zz}\right\}\phi = 0$$

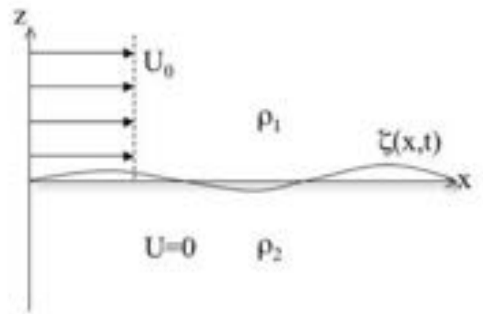
$$(U-c)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{N^2}{(U-c)} - U_{zz}\right\}\phi = 0$$

Note that $c_{ph} = c - U$ is the phase velocity within the moving frame, and $\Omega = ck - Uk = \omega - Uk$ the Doppler shifted or intrinsic frequency.
It can be shown that for stability (see e.g. Drazin & Reid p327) :

$$Ri > 1/4$$

with Richardson number (also Ri) $= \frac{N^2}{(\partial U/\partial z)^2}$, N the Brunt Väisälä frequency)

INTERMEZZO INTERNAL WAVES



What happens when the fluid is unstably stratified ?
i.e $\rho_1 > \rho_2$

intermezzo internal waves $Im(c) = 0$

In case the velocity is zero $U = 0$ we obtain the relation for the perturbation in vertical velocity w' (2D) :

$$w'_{zz} + \left\{ \frac{N^2}{c^2} - k^2 \right\} w' = 0$$

with say $kw' = 0$ at $z = 0$ and H.

Internal waves may exist for $N^2 \neq 0$.
Consider the simplest case : $\bar{\rho} = \rho_0 exp(-z/H)$,
then $N^2 = g/H = constant$. Then

$$c^2 = \frac{N^2}{(k^2 + n^2\pi^2/H^2)}$$
$$w = sin\{n\pi(z/H)\} \text{ for } n = 1, 2...$$

represent a discrete spectrum of internal gravity waves (stable or unstable depending on the sign of N^2).

intermezzo internal waves

With perturbations of the form in the (x, z) plane :
$w = \hat{w} \exp[i(kx + mz - \omega t)]$ the dispersion relation for ω is (only waves so that $Re(\omega) \neq 0$ and $Im(\omega) = 0$)

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2}$$

The phase and group velocities are

$$c_{px} = \frac{\omega}{k} = \pm \frac{N}{(k^2 + m^2)^{1/2}} \qquad c_{pz} = \frac{\omega}{m} = \pm \frac{Nk/m}{(k^2 + m^2)^{1/2}}$$
$$c_{gx} = \frac{\partial \omega}{\partial k} = \pm \frac{Nm^2}{(k^2 + m^2)^{3/2}} \qquad c_{gz} = \frac{\partial \omega}{\partial m} = \mp \frac{Nmk}{(k^2 + m^2)^{3/2}}$$

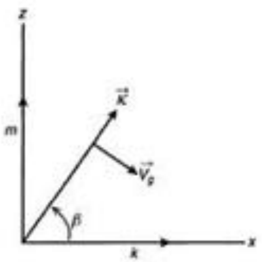
and so there is dispersion in x and z direction.

intermezzo internal waves

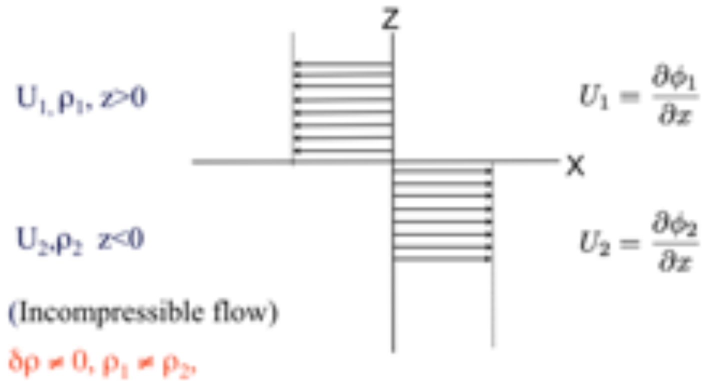
For a downward propagating internal wave with upward propagating wave energy we have :

$$\vec{k} = k\vec{e_x} - m\vec{e_z}$$
$$\vec{c_g} = \frac{Nm}{(k^2 + m^2)^{1/2}} (m\vec{e_x} + k\vec{e_z})$$

we have $\vec{c_g}.\vec{k} = \frac{Nm}{(k^2+m^2)^{1/2}} (km - mk) = 0$



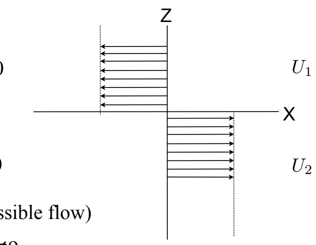
CLASS EXERCISE



calculate the dispersion relation using Bernoulli

EXERCISE- Instability of a vortex sheet

U_1, ρ_1, z>0



$$U_1 = \frac{\partial \phi_1}{\partial x}$$

U_2, ρ_2 z<0

$$U_2 = \frac{\partial \phi_2}{\partial x}$$

(Incompressible flow)

$\delta\rho \neq 0$, $\rho_1 \neq \rho_2$,

$$U_{1,2} = \frac{(U_1 + U_2)}{2} \pm \frac{U_1 - U_2}{2} = C \pm \frac{U}{2}$$

The *frame is moving with speed C* (so that $U_1 = -U_2 = U/2$),

Perturbations φ' on the basic flow, so that $\vec{U} = \left(U_i + \frac{\partial \phi'_i}{\partial x}, \frac{\partial \phi'_i}{\partial z}\right)$

The flow satisfies the Bernoulli equation

$$\frac{\partial \phi_i}{\partial t} + \frac{1}{2}\nabla\phi_i{}^2 + \frac{P_i}{\rho_i} + gz = C_i$$

BOUNDARY CONDITIONS TO CALCULATE THE DISPERSION RELATION

1) u=∇φ, and ∇.u= so that $\nabla\phi_1 = 0$, $\nabla\phi_2 = 0$ i.e $\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0$

Perturbations φ'⟶0 for z ⟶ ±∞

2) Kinematic interface condition with $w_1 = w_2$ and $w_i = \frac{\partial \phi_i}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi_i}{\partial x}\frac{\partial \zeta}{\partial x}$

3) Dynamic interface condition, pressure (normal to the interface) is continuous, i.e. $P_1 = P_2$

$$(C_1 - \frac{\partial \phi_1}{\partial t} - \frac{1}{2}\nabla\phi_1{}^2 - gz)\rho_1 = (C_2 - \frac{\partial \phi_2}{\partial t} - \frac{1}{2}\nabla\phi_2{}^2 - gz)\rho_2 \text{ (at } z = \zeta)$$

This condition (at z=0) should be satisfied also by the basic flow:

$$\rho_1(C_1 - \frac{1}{2}U_1^2) = \rho_2(C_2 - \frac{1}{2}U_2^2) \qquad \text{(note } \nabla\phi = U_i + \frac{\partial \phi'_i}{\partial x})$$

For $\rho_1 = \rho_2$ and $U_1 = -U_2 : C_1 = C_2$

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PERTURBATIONS OF THE FORM $(\zeta', \phi'_1, \phi'_2) = (\hat{\zeta}, \hat{\phi}_1, \hat{\phi}_2)(z)e^{ikx+\sigma t}$

1) $\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0$ gives $-k^2\hat{\phi}_i + \frac{\partial^2 \hat{\phi}_i}{\partial z^2} = 0$

Solutions are of the form $\hat{\phi}_i = A_i e^{-kz} + B_i e^{kz}$

With the condition φ'⟶0 for z ⟶ ±∞ we obtain $\hat{\phi}_1 = B_1 e^{kz}$ and $\hat{\phi}_2 = A_2 e^{-kz}$

2) Kinematic interface condition $kA_2 = -(\sigma + ikU_2)\zeta$ and $kB_1 = (\sigma + ikU_1)\zeta$

3) After linearisation of the Bernoulli equation we obtain (after subtraction of the basic state) (note again: $(U_i + \frac{\partial \phi'_i}{\partial x})^2 = (U_i^2 + 2U_i\frac{\partial \phi'_i}{\partial x} + \left(\frac{\partial \phi'}{\partial x}\right)^2)$)

$$\rho_1(\sigma + ikU_1\hat{\phi}_1 + g\hat{\zeta}) = \rho_2(\sigma + ikU_2\hat{\phi}_2 + g\hat{\zeta})$$

Substitute the expressions for ˆφ and ˆζ above in 3)
to obtain the DISPERSION relation for *σ(k, U, gΔρ)*

$$\rho_1[kg + (\sigma + ikU_1)^2] = \rho_2[kg - (\sigma + ikU_2)^2]$$

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We discuss the different instabilities later in this course.