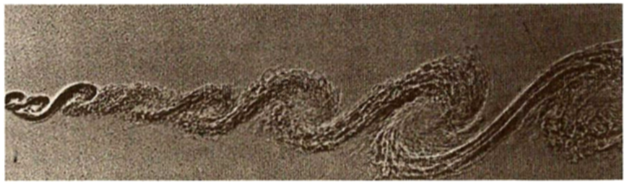
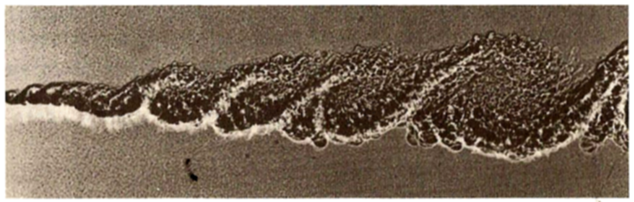
The mixing layer

shear layer thickness $\delta(x) = \frac{(U_1+U_2)}{(dU/dy)_{max}}$ ~√x

δ increases with x by diffusion ; vortex roll-up and vortex merging.

δ becomes linear in x far downstream.





mixing layer: continuous profile with y

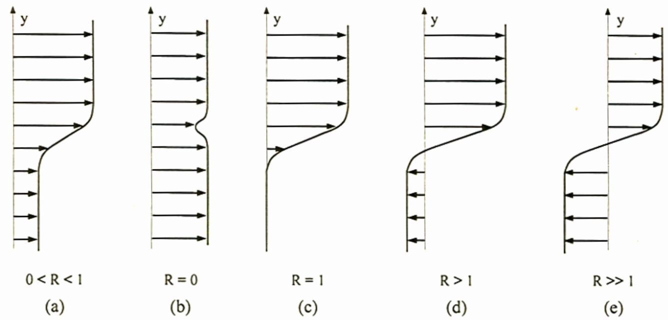


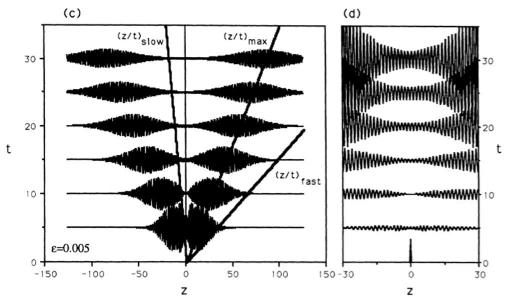
Fig. 2.6. Typical mixing layer velocity profiles as a function of velocity ratio R when $U_1 \geq |U_2|$: (a) $0 < R < 1$; (b) $R = 0$; (c) $R = 1$; (d) $R > 1$; (e) $R \gg 1$.

R dilatation parameter in X direction (growth rate versus convection velocity)

R = 1 : lower stream at rest; increase in layer thickness proportional to R
R<<1 : convection velocity >> growth ΔU
R > 1 : Counter flow mixing layers
R>>1 : well defined KH billows at one position
NOISE: mixing layer is a noise amplifier.

Godreche & Manneville 1998

$|R| << 1$ weak shear ; simple linear relation between spatial and temporal development of instability
$|R| \geq 1$ complex relation between spatial and temporal development of instability.



Convective versus absolute instability

Control numbers are :

$$\delta = \frac{(U_1 - U_2)}{(d^2U/dy^2)_{max}} \quad , \; Re = \frac{(U_1 - U_2)\delta(0)}{\nu}$$

and velocity ratio $R = \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$

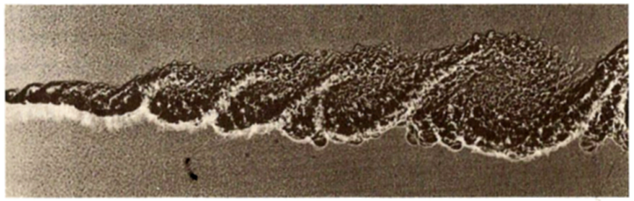
R=0 : no net shear (e.g. wake behind a flat plate)

δ increases proportionally with shear intensity R (growth rate)
R«1 slow streamwise development
$R \approx 1$ or $R \geq 1$, roll-up and merging occur closer to x=0.

Strouhal number describing the characteristic frequency of vortices

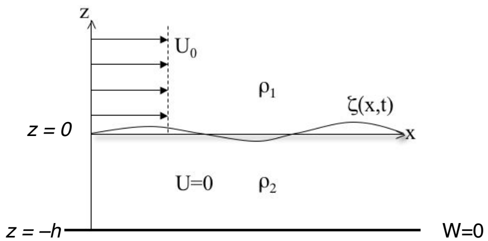
$$St_n = \frac{f_n \delta(0)}{\bar{U}} \approx 0.03$$

f_n is the natural vortex frequency in the wake



EXERCISE

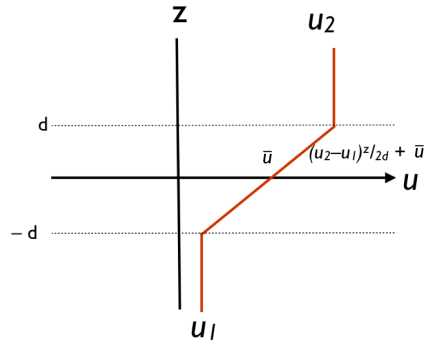
Laminar basic flow ;
with layers 1 and 2 of different density



Viscous effets are considered negligible and the fluid is incompressible.
This flow satisfies the Euler equations, continuity and hydrostatic balance.

STEP WISE VELOCITY PROFILE

Shear layer with non-zero thickness (Rayleigh 1869)



We take for simplicity $\bar{x} \rightarrow (x, z)$ (ρ=constant)
pressure $p = p(z)$
Basic flow + perturbations : $(U_0 + u, w, P_0 + \delta p)$

- write down adapted (2D) Euler equations and basic state
- derive perturbation equations
- define the form of the perturbation
- Substitute and obtain a PDE for w at z=0

substitute in the Euler equations :

$$\frac{\partial u}{\partial t} + U_0\frac{\partial u}{\partial x} + w\frac{\partial U_0}{\partial z} = -\frac{1}{\rho}\frac{\partial \delta p}{\partial x}$$

$$\frac{\partial w}{\partial t} + U_0\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial \cancel{P_0} + \delta p}{\partial z} - \cancel{g}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Substitute perturbations : $v(x, z, t) = \hat{v}(z) exp\{i(kx + \omega t)\}$

$$i(\omega + kU_0)u + w\frac{\partial U_0}{\partial z} = -\frac{ik}{\rho}\delta p$$

$$i(\omega + kU_0)w = -\frac{1}{\rho}\frac{\partial}{\partial z}\delta p$$

$$u = \frac{i}{k}\frac{\partial w}{\partial z}$$

boundary conditions

Reduce variables to obtain a partial differential equation in z (eliminate u with i and iii)

$$-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = -\frac{ik^2}{\rho}\delta p \qquad (4)$$

eliminate δp to obtain a single equation in w

$$\frac{\partial}{\partial z}\left[-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z}\right] = k^2(\omega + kU_0)w \qquad (5)$$

The *kinematic boundary condition* imposes that w is continuous across the interface :

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} w\, dz = 0$$

Applying this to equation (5) yields :

$$-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = 0 \qquad (6)$$

$\int (4)/\partial z$ is equal to the pressure gradient; (6) implies $\delta p_1 - \delta p_2 = 0$ so that also the *dynamic boundary condition* is satisfied.

Show that in regions where $\frac{\partial U_0}{\partial z} = 0$ we have $\frac{\partial^2 w}{\partial z^2} - k^2 w = 0$ ...
In the three regions we have :

$z > d$	$w = A_+ e^{-kz}$
$-d < z < d$	$w = A_0 e^{-kz} + B_0 e^{kz}$
$z < -d$	$w = A_- e^{kz}$

with the constants A_-, A_+, A_0 and B_0 to determine with the continuity accros the interface, i.e.
1) Kinematic boundary condition : continuity of w at $\pm d$
2) Continuity of pressure gives :

$$-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = 0$$

(suppose $U_2 = U$ and $U_1 = -U$ and the relation found with *1)*)

Continuity of w at $\pm d$:

$$+d : A_+ e^{-kd} = A_0 e^{-kd} + B_0 e^{+kd}$$
$$-d : A_- e^{-kd} = A_0 e^{kd} + B_0 e^{-kd}$$

gives with continuity of $-(\omega + kU_0)\frac{\partial w}{\partial z} + kw\frac{\partial U_0}{\partial z} = 0$:

$$+d : 2(\omega + kU)B_0 e^{kd} - \frac{U}{d}(A_0 e^{-kd} + B_0 e^{kd}) = 0$$
$$-d : 2(\omega - kU)A_0 e^{kd} + \frac{U}{d}(A_0 e^{kd} + B_0 e^{-kd}) = 0$$

Elimination of $\frac{A_0}{B_0}$ yields the dispersion relation for ω (Rayleigh 1896 vol11, p 393 and Drazin p 146 :

$$\omega^2 = \frac{U^2}{4d^2}\left[(1 - 2kd)^2 - e^{-4kd}\right]$$

since $\sim \exp[i(kx + \omega t)]$ instability for $i\omega > 0$.

Simplify the dispersion relation $\alpha = 2kd$ and $\Omega = \omega/(2kU)$
Since $U_1 = -U_2 = -U$, the phase velocity is $c = \omega/k$ (in case there is a mean velocity, it increases the phase velocity)

$$4\alpha^2\Omega^2 = (1 - \alpha)^2 - e^{-2\alpha}$$

so that :

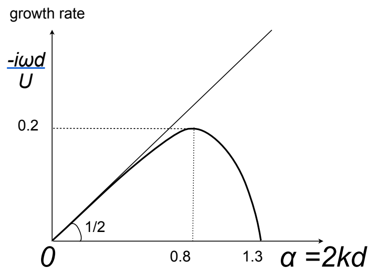
$$\Omega^2 = 1/4\frac{\left[(1 - \alpha)^2 - e^{-2\alpha}\right]}{\alpha^2}$$

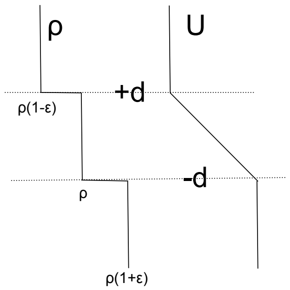
deduce Kelvin Helmholtz instability, i.e. $d \to 0$,

$$\omega = ikU$$

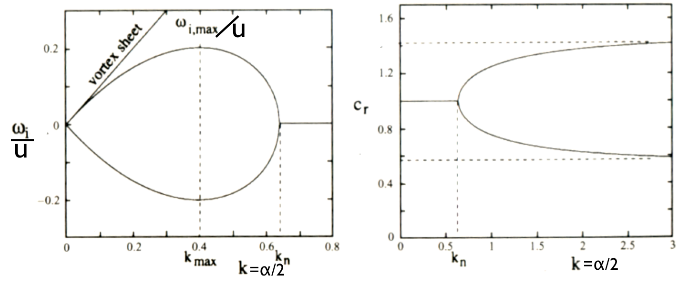
$$\omega^2 = \frac{U^2}{4d^2}\left[(1 - 2kd)^2 - e^{-4kd}\right]$$

Kelvin Helmholtz





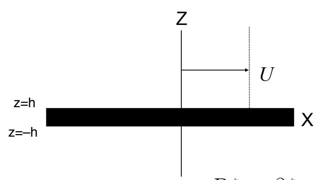
$\Omega^2 = \frac{1}{4\alpha^2}\left[(1-\alpha)^2 - e^{-2\alpha}\right]$ and $\Omega = \omega/(2kU)$ and $c_r = \omega_r/k$



Large wave lengths (small k) do not see the thickness of the interface and are unstable as KH
Short wave lengths are stabilized (large k), they are within the shear layer.

Vorticity layer instability

Consider the instability of the vorticity layer at the interface (2D)



vorticity ζ layer at z=0

Vorticity from Euler equations: $\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + U.\nabla\zeta = 0 \qquad \zeta = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$

Continuity $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

Basic flow is *U* in *x*-direction, i.e. (*U,0*)
Perturbation (*u',w'*) ⟶ (*u,w)=(U+u', w')*

$$\zeta = \frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} - \frac{\partial U}{\partial z}$$

$$\frac{\partial \zeta}{\partial t} + (U+u')\frac{\partial \zeta}{\partial x} + w'\frac{\partial \zeta'}{\partial z} = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

Consider the instability of the vorticity layer at the interface (2D)

Linearise, neglect terms of second order

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{\partial^2 U}{\partial z^2}w' = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

Consider a perturbation of the form $u', w' \longrightarrow (u'(z), w'(z))\, e^{ikx + i\sigma t}$,

$$i(\sigma + kU)\left(ikw' - \frac{\partial u'}{\partial z}\right) - \frac{d^2U}{dz^2}w' = 0$$

$$iku' + \frac{\partial w'}{\partial z} = 0$$

Eliminate *u'* to find *THE* ordinary differential equation in *z* to solve:

$$(\sigma + kU)\left(\frac{\partial^2 w'}{\partial z^2} - k^2 w'\right) - \frac{d^2U}{dz^2}kw' = 0$$

Since for z=0, (the region of interest), *dU/dz* is discontinuous we have to replace this differential with the difference Δ across the two layers:

$$\lim_{\Delta\to 0}\int_{-\Delta/2}^{\Delta/2}(\sigma + kU)\left(\frac{\partial^2 w'}{\partial z^2} - k^2 w'\right) - \frac{d^2U}{dz^2}kw' \; dz =$$

$$(\sigma + kU)\Delta\frac{\partial w'}{\partial z} - \Delta\frac{\partial U}{\partial z}kw' = 0$$

Note: we have used here w' continuous across the interface

Move with the fluid, i.e. **u** = U/2 for z > h
and **u** = – U/2 for z < –h

For z>h and z<–h *d²U/dz²=0*, we have (as before)

$$\frac{\partial^2 w'}{\partial z^2} - k^2 w' = 0$$

So that for we obtain for the different layers (as before):

$$w' = Ae^{-kz} \qquad \text{for z>h}$$
$$w' = Be^{-kz} + Ce^{kz} \qquad \text{for -h<z<h}$$
$$w' = De^{kz} \qquad \text{for z<-h}$$

Continuity of *w'* at z>h and z<–h gives then

$$Ae^{-kh} = Be^{-kh} + Ce^{kh}$$
$$De^{-kh} = Be^{kh} + Ce^{-kh}$$

and the relation $(\sigma + kU)\Delta\frac{\partial w'}{\partial z} - \Delta\frac{\partial U}{\partial z}kw' = 0$ gives with ***u***= ± *U/2*

$$2(\sigma + k\mathbf{u})Ce^{kh} - \frac{\mathbf{u}}{h}(Be^{-kh} + Ce^{kh}) = 0$$
$$2(\sigma - k\mathbf{u})Be^{kh} + \frac{\mathbf{u}}{h}(Be^{kh} + Ce^{-kh}) = 0$$

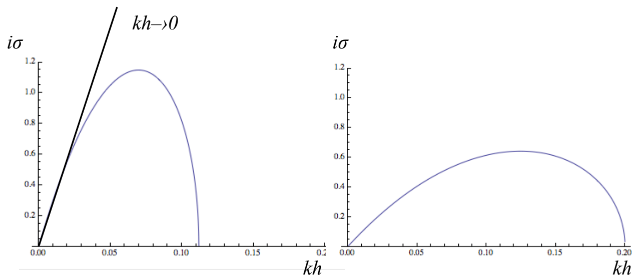
eliminate *B* and *C* gives then …

$$\sigma^2 = \frac{\mathbf{u}^2}{4h^2}\left[(2kh-1)^2 - e^{-4kh}\right]$$

in the limit of $kh \to 0$ $\sigma^2 = -k^2\mathbf{u}^2$ with $u', w' \sim e^{ikx + i\sigma t}$, we note that $i\sigma > 0 \longrightarrow$ growth !
Same as the KH interface from above.

For large values of *kh* shear layer thickness decreases the growth σ
$\sigma^2 = +k^2\mathbf{u}^2$ so that $\sigma = \pm k\mathbf{u}$; Since Im(σ)=0, stability

$$\sigma^2 = \frac{\mathbf{u}^2}{4h^2}\left[(2kh-1)^2 - e^{-4kh}\right]$$



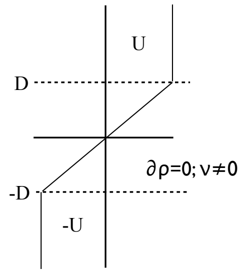
→ *h larger*

cut off wave number

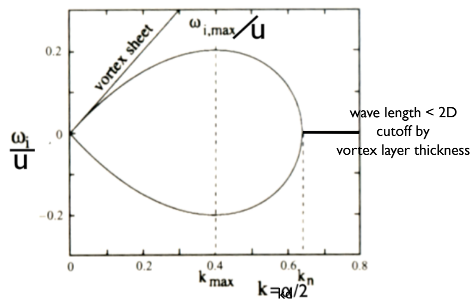
(same result as above)

Viscosity: diffusive effects !

Effect of viscosity on the instability of a shear layer

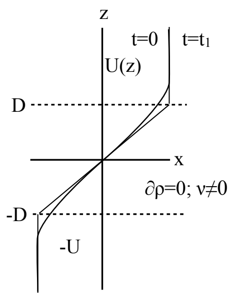


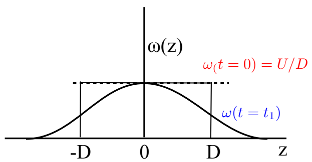
for Re=Ud/ν –> ∞
instability for a band of wavelengths 0<2kd<1.3
maximum growth at 2kd=0.8



What is the effect of viscosity on :
- growth rate ?
- wavenumber ?

viscous spreading of the shear layer





$$\left.\begin{aligned}\omega &= \nabla \times U \\ U &= U(z)\end{aligned}\right\} \rightarrow \omega = (0, \frac{\partial U(z)}{\partial z}, 0)$$

$$\frac{\partial \bar{\omega}}{\partial t} = \nabla \times (\bar{u} \times \bar{\omega}) + \nu \nabla^2 \bar{\omega} \qquad \nabla \times (\bar{u} \times \bar{\omega}) = \bar{u}(\nabla.\bar{\omega}) - \bar{\omega}(\nabla.\bar{u}) + (\bar{\omega}.\nabla)\bar{u} - (\bar{u}.\nabla)\bar{\omega}$$

$$\frac{\partial \bar{\omega}}{\partial t} = -\bar{u}.\nabla\bar{\omega} + \bar{\omega}.\nabla\bar{u} + \nu \nabla^2 \bar{\omega}$$

$\bar{u} \perp \bar{\omega} \rightarrow \bar{\omega}.\nabla\bar{u} = 0$

uni directional flow U(z) $\rightarrow \bar{u}.\nabla\bar{\omega} = 0$

$$\rightarrow \frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \qquad \omega = \omega_y$$

vorticity diffusion equation

The viscous spreading of the shear layer

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \text{ here } \omega = \omega_y$$

For a *thin shear layer* of thickness δ(y) and amplitude U

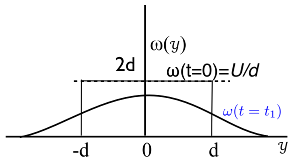
$$\omega(y, t = 0) = U\delta(y)$$

the *solution* of the diffusion equation is (see Batchelor 1969)

$$\omega(y, t) = \frac{U}{2\sqrt{\pi \nu t}} e^{-y^2/4\nu t}$$

for a *shear layer from y = –d to +d* $\omega(y, t) = \int_{-d}^{+d} \frac{\omega(y - y', t)}{2d} dy$

the solution is $\omega(y, t) = \frac{U}{2d}[erf\left(\frac{y + d}{\sqrt{4\nu t}}\right) - erf\left(\frac{y - d}{\sqrt{4\nu t}}\right)]$ with $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$



$$\omega(y, t \rightarrow \infty) = 0 \text{ but } \int_{-\infty}^{+\infty} \omega(y, t) dy = 2U$$

=>

- velocity jump across the layer is maintained
- vorticity decays by viscous damping.

Determine the speed of the spreading =>

Thickness of the diffusing shear layer.

The standard deviation of the vorticity distribution at t=0 is

$$\sigma^2 = \frac{1}{2u}\int_{-\infty}^{+\infty} y^2\omega(y,t)dy$$

this is generally smaller than the real distribution (here 2D) so rescale:

note d=D, u=U, Δ=D at t=0

if $\frac{\sigma^2}{\Delta^2} = \frac{1}{2u}\int_{-\infty}^{+\infty} y^2\omega(y,t)dy = \frac{1}{a}$ then is $\Delta^2 = \frac{a}{2u}\int_{-\infty}^{+\infty} y^2\omega(y,t)dy$

and $a = \left(\frac{\Delta}{\sigma}\right)^2 = \left(\frac{D}{\sigma}\right)^2_{t=0}$

For a linear velocity profile a=3 (at t=0), the integral yields

$$\Delta^2 = D^2 + \delta^2 \qquad \delta = \frac{3}{2}\sqrt{4\nu t}$$

The spreading of the vorticity distribution can then be written as

$$\boxed{\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{2\delta}{D^2+\delta^2}\frac{d\delta}{dt}}$$

two cases. 1) weak viscous spreading δ/D<<1 an
2) thin layer with strong viscous effects, i.e. δ/D>>1

If δ/D<<1 viscous effects are small at t=0, initial thickness is large

$$\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{2\delta}{D^2+\delta^2}\frac{d\delta}{dt} \approx \frac{2\delta}{D^2}\frac{d\delta}{dt} \approx \frac{2\nu}{D^2} \quad \text{=constant in time}$$

$$\delta \sim \sqrt{\nu t}$$

If δ/D>>1 t=0, thin layer with strong viscous effects (Δ≈δ)

$$\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{2\delta}{D^2+\delta^2}\frac{d\delta}{dt} \approx \frac{1}{\delta}\frac{d\delta}{dt} = \frac{1}{t}$$

Now compare this with the growth rate of the instability *(Re= Real part)*

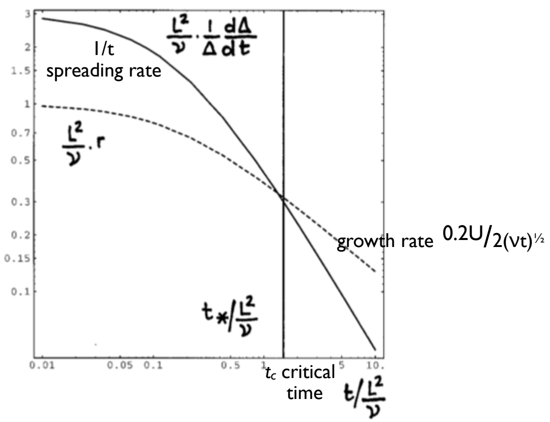
$$Re(-i\sigma) = \frac{0.2U}{\Delta}$$ which is the maximum growth rate for the inviscid case

this growth rate is affected by viscosity due to increase in thickness Δ,

- in case δ/D<<1 Δ=D and the growth rate, 0.2 U/D, is not affected.
- In case Δ≈δ the spreading of the viscous layer is faster than the growth of the instability.

$$Re(-i\sigma) = \frac{0.2U}{D} \approx \frac{0.2U}{\delta} = \frac{0.2U}{2\sqrt{\nu t}} \quad \text{and spreading of layer is} \quad \frac{1}{\delta}\frac{d\delta}{dt} = \frac{1}{t}$$

spreading rate of the vorticity



At **critical time** $t_c = 100\ \nu/U^2$ -> $\delta_c = (\nu t_c)^{1/2} = 10\nu/U$

==> Critical time depend on Reynolds number UD/ν

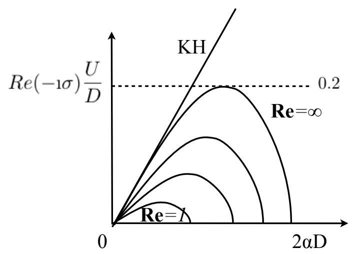
Critical time t_c as a function of Reynolds number **Re**

with $\delta_c = (\nu t_c)^{1/2} = 10\nu/U$ one can write **Re** = UD/ν =D/(δ_c/10) so that δ/D=10/Re

For the growth rate we obtain: $Re(-\imath\sigma) = \frac{0.2U}{\sqrt{D^2+\delta^2}} = \frac{U}{D}\frac{0.2}{\sqrt{1+\delta^2/D^2}}$

so that the non-dimensional growth is

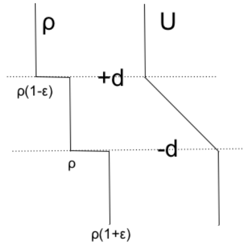
$$Re(-\imath\sigma)\frac{U}{D} = \frac{0.2}{\sqrt{1+(10/\mathbf{Re})^2}}$$



Villermaux 1998 Phys. of Fluids
Betchov Szewczyk Phys. of Fluids 1963

With density distributions I

(see Chandrasekhar 1961, p488-489)



$z > d$	$\rho = \rho_0(1-\epsilon)$
$-d < z < d$	$\rho = \rho_0$
$z < -d$	$\rho = \rho_0(1+\epsilon)$

the dispersion relation reads :

$$e^{-2\alpha} = \left[1 - \frac{\alpha(\Omega+1)^2}{J+(\Omega+1)+\epsilon\alpha/2(\Omega+1)^2}\right]\left[1 - \frac{\alpha(\Omega-1)^2}{J-(\Omega-1)-\epsilon\alpha/2(\Omega-1)^2}\right]$$

with $\Omega = \omega/(kU)$ and J the Richardson number :

$$J = \frac{\epsilon g k}{2U^2 k} \sim \frac{g\Delta\rho/2d}{\rho(dU/dz)^2}$$

For stability $Re(\Omega^2) > 0$. Unstable when $Re(\Omega^2) < 0$ i.e. when

$$\frac{k}{1+e^{-k}} < J+1 < \frac{k}{1-e^{-k}}$$

The Richardson number represents the ratio between the kinetic energy of relative motion $(\frac{\partial U}{\partial z})^2$ and the work that must be done to overcome the restoring buoyancy force.

(see exercise on particle displacement of lecture 1 ; note that in this exercise the Boussinesq approximation is used by assuming that $\Delta\rho U$ is small). The results for instability is :

$$Ri(=\boldsymbol{J}) = \frac{-g}{\bar{\rho}}\frac{d\rho/dz}{(dU/dz)^2} = \frac{\text{buoyancy force}}{\text{inertia force}} < \frac{1}{4}$$

Exercise :

Consider a basic flow with velocity profile $U(z)$ and density distribution $\rho(z)$ and neglect viscous effects. Derive the dispersion relation (Taylor-Goldstein equation).

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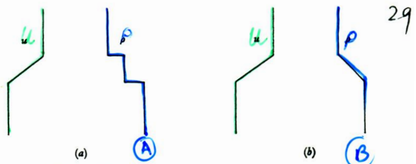


Fig. 4.2. Velocity and density profiles used in the first interfacial stability calculations.

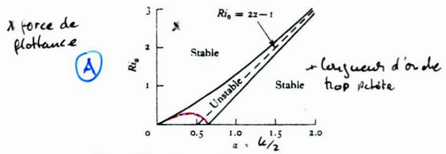


Fig. 4.3. The stability characteristics of a shear layer corresponding to fig. 4.2a. Waves which can grow on the density discontinuities contribute significantly to the instability.

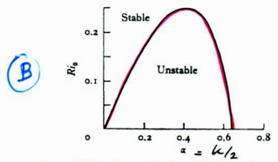
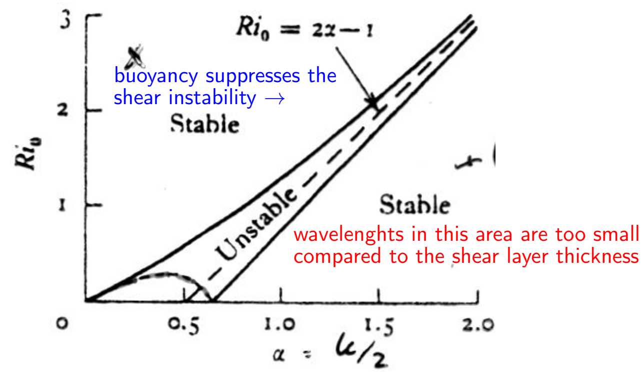


Fig. 4.4. The stability characteristics of a shear layer corresponding to fig. 4.2b. Note the difference in scales, and the very much smaller region of instability compared to fig. 4.3. (The stability boundary in fig. 4.4 has been added to 4.3 for comparison, as the dotted line.)



instability for $\frac{k}{1+e^{-k}} < J+1 < \frac{k}{1-e^{-k}}$

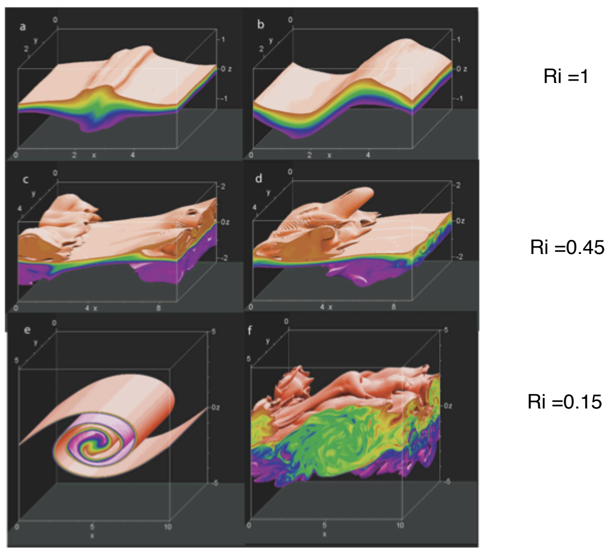
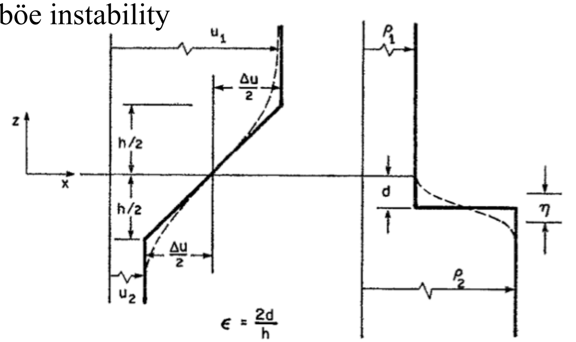


FIG. 3. Snapshots illustrating spatial structures of three mixing events. Each event is shown at two successive times. Colors represent the buoyancy field; values range from 0.4Δb (red) to −0.4Δb (purple). Values outside this range are rendered transparent. (top) Run 1, Ri_0 = 1, (a) t = 234.3, (b) t = 237.8. (middle) Run 3, Ri_0 = 0.45, (c) t = 143.4, (d) t = 152.9. (bottom) Run 5, Ri_0 = 0.15, (e) t = 35.40, (f) t = 112.5.

Hölmböe instability

Hölmböe instability



We take for simplicity again $\bar{x} \rightarrow (x, z)$
Density interface at the level $z = -d$; thickness η and shear layer with total thickness h (Lawrence et al Phys. of Fluids 1991).

We first consider waves $\overline{U} = 0$ and then instabilities for $\epsilon = 0$.

Hölmböe instability : the mechanism consider Rayleigh's 1896 shear flow for $U_{z=0} = 0$

Mutually interacting waves:

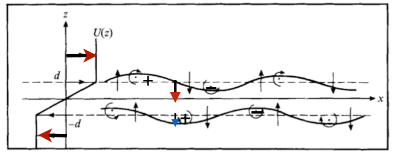
$$w_1 = i\omega\zeta_1 + ikU \rightarrow \frac{w_1}{kU} = i\frac{c}{U}\zeta_1 + i$$

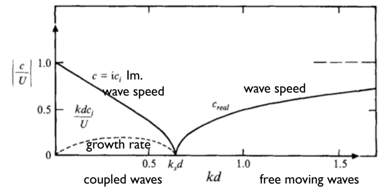
$$w_2 = i\omega\zeta_2 - ikU \rightarrow \frac{w_2}{kU} = i\frac{c}{U}\zeta_2 - i$$

$$\left(\frac{\omega/k}{U}\right)^2 = \left(\frac{c}{U}\right)^2 = \frac{(1-2kd)^2 - e^{-4kd}}{(2kd)^2}$$

instability when phase shift is λ/4
damping for phase shift λ/2

The shift is due to the Doppler effect on the oscillatory motion (c_r) -> resonance!





two types of instability:
1) stationary (i.e. non-oscillatory),
for p'~ $exp\{ik(x\text{-}ct)\}$ imaginary part =0 (c_r=0)

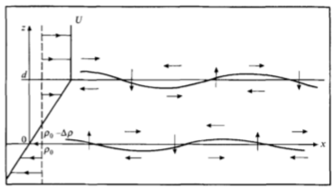
2) travelling wave on the vorticity interface and a standing wave on the density interface.

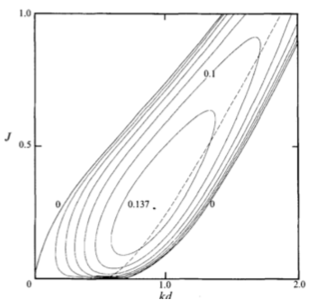
$$(c^2 - c_1^2)(c - (U_0 - c_2)) + c_1^2 c_2 e^{-kd} = 0$$

$$c_1^2 = g'/2k \text{ and } c_2 = U_0/2kd$$

same mechanism

The phase shift is due to the Doppler effect on the oscillatory motion

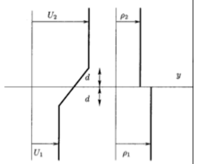




Holmboe (1962),
Baines and Mitsudera (1994)

Suppose step-profile, symmetric interface ($\epsilon = 0$) in

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{JN^2}{(U-c)^2} - \frac{U''}{(U-c)}\right\}\phi = 0.$$



Then the dispersion relation reads

$$D(k, \omega, J, a) = (\omega - ak)^4 + n_2 k^2(\omega - ak) + n_0 k^4 = 0$$

where $a = U_{mean}/\Delta U$

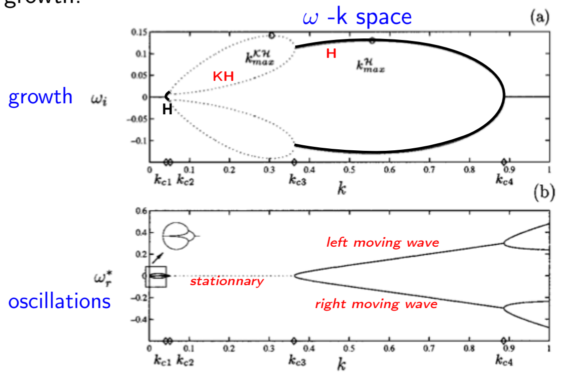
$$n_2 = \frac{-J}{sk} + \frac{e^{-4sk} - (2sk-1)^2}{4k^2} \text{ and } n_0 = \frac{J}{sk} + \frac{(e^{-2sk} + 2sk - 1)^2}{4k^2}$$

with $s = sgn(k_r)$. The roots are then (Ortiz et al POF 2002) :

$$\omega(k) = ak \pm \left\{\frac{-n^2k^2 \pm \Delta^{1/2}}{2}\right\}^{1/2} \text{ and } \Delta = (n_2k^2)^2 - 4n_0k^4$$

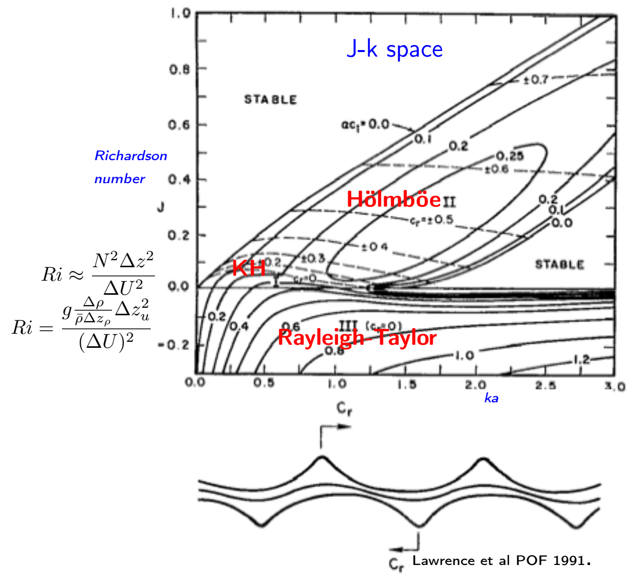
$$\omega(k) = ak \pm \left\{ \frac{-n^2 k^2 \pm \Delta^{1/2}}{2} \right\}^{1/2} \quad \text{and } \Delta = (n_2 k^2)^2 - 4 n_0 k^4$$

advection with speed $a = U_m/\Delta U k$ results in Doppler shift . To move with the local mean flow we should take $\omega_r^* = \omega_r - ak$ where $\omega = \omega_r + i\omega_i$; ω_r representing the oscillatory part and ω_i the growth.



four neutral waves:
2 propagate to the right
(ω>0)2 to the left (ω<0)

Ortiz et al, Phys of Fluids 2002



Lawrence et al POF 1991.

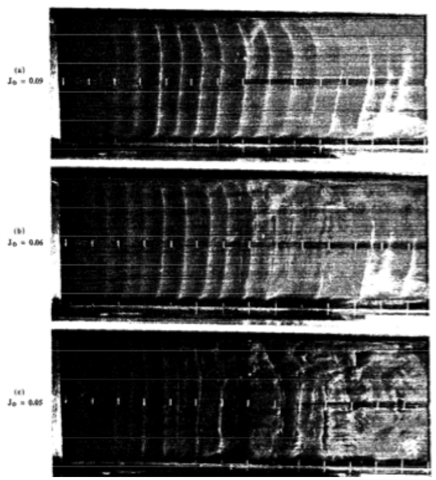
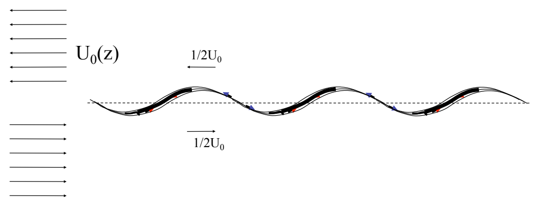


FIG. 7. Plan views of experiments with $J_0 = 0.09$, 0.06, and 0.05, respectively. Flow is left to right. The markings on the bottom of the flume are spaced at 5 cm intervals. The trailing edge of the splitter plate is visible on the left-hand side of each photograph.

Lawrence, Browand & Redekop 1991

see movies ...

KH instability: vorticity layer rolls up at the interface



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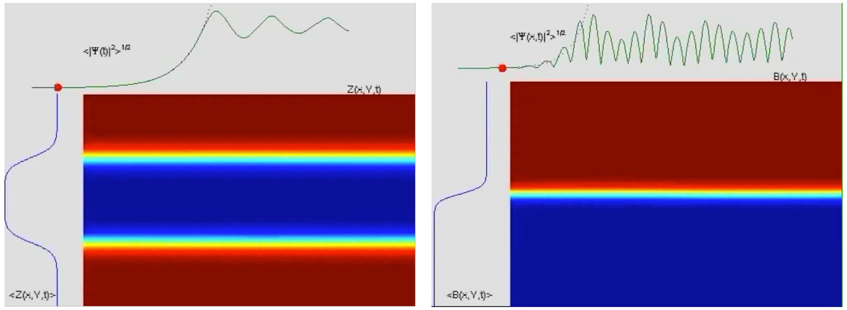
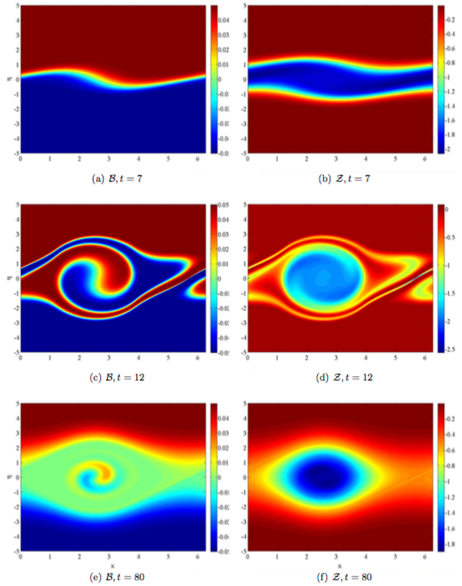


Neil Balmforth 2010, Woodshole



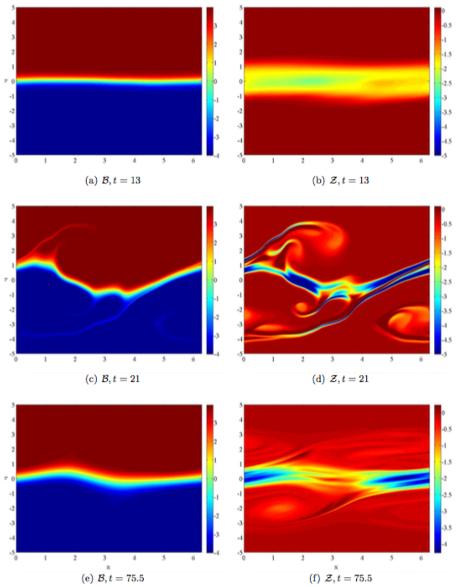
(a) $\mathcal{B}, t = 7$ (b) $\mathcal{Z}, t = 7$

(c) $\mathcal{B}, t = 12$ (d) $\mathcal{Z}, t = 12$

(e) $\mathcal{B}, t = 80$ (f) $\mathcal{Z}, t = 80$

'igure 9: Vorticity and buoyancy fields at different instants of time for stratified KH instability, $f = 2, d = 0.1, k = 1, \Delta = 1/3, \eta_0 = 1$.

Roy Anubhab 2010, Woodshole



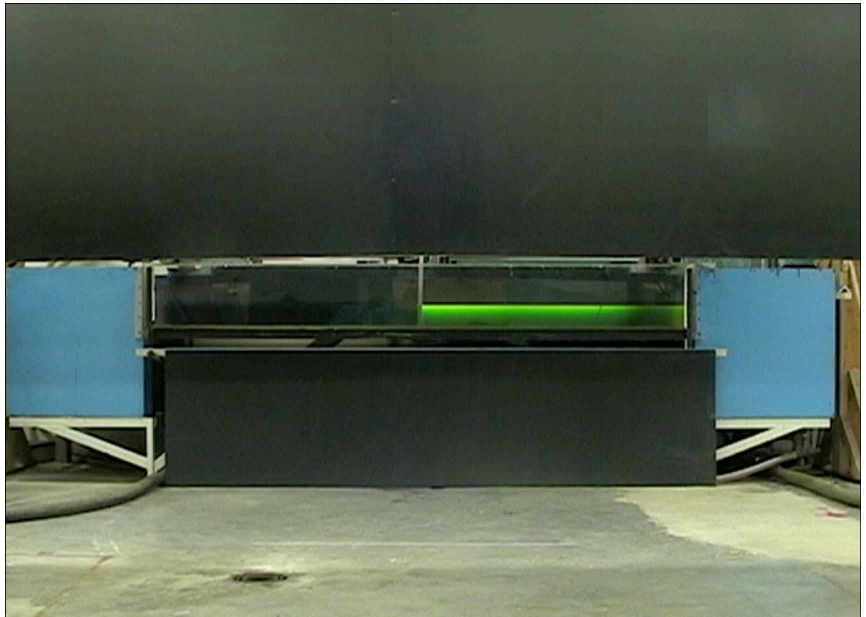
(a) $\mathcal{B}, t = 13$ (b) $\mathcal{Z}, t = 13$

(c) $\mathcal{B}, t = 21$ (d) $\mathcal{Z}, t = 21$

(e) $\mathcal{B}, t = 75.5$ (f) $\mathcal{Z}, t = 75.5$

;ure 11: Vorticity and buoyancy fields at different instants of time for pure Holmbo tability, $f = 2, d = 8, k = 1, \Delta = 1/3, \eta_0 = 1$.

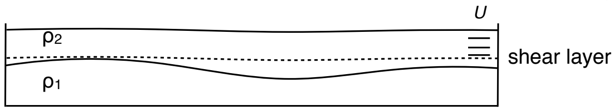
Roy Anubhab 2010, Woodshole

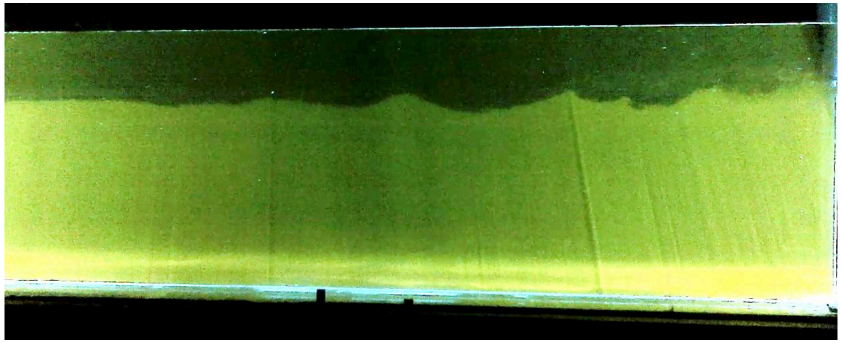


Measurements on KH instability

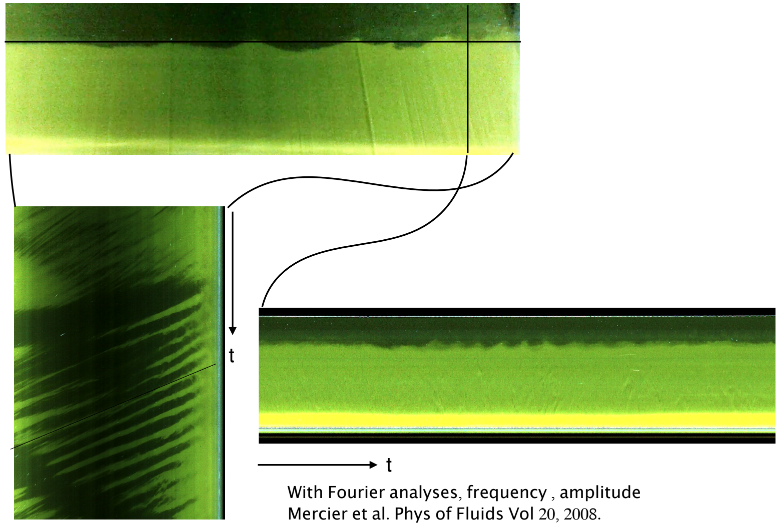
$$Ri \approx \frac{N^2 \Delta z^2}{\Delta U^2} \approx \frac{1}{4}$$

$$Ri = J\ R\ = \frac{g'\ \Delta z_u}{\Delta U^2}\left(\frac{\Delta z_u}{\Delta z_\rho}\right)$$



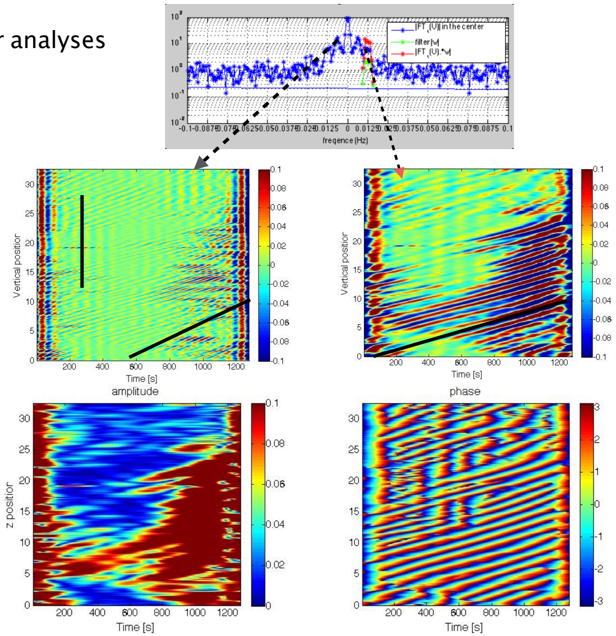


Space time diagram (also Hövmöller diagram)



With Fourier analyses, frequency , amplitude
Mercier et al. Phys of Fluids Vol 20, 2008.

Fourier analyses



Continuous velocity profiles.

Exercise :
Consider a basic flow with velocity profile $U(z)$ and density distribution $\rho(z)$ and neglect viscous effects. Derive the dispersion relation (Taylor-Goldstein equation).

The Taylor-Goldstein equation

- Parallel flow U(z) $[U + u', v', w']$ and stratification N.
- Euler Equations, viscosity $\nu = 0$
- Squires theorem $(\nu = 0)$: $3D \rightarrow 2D$, stronger growth for 2D than 3D (see later)
- linearize, define a stream function $u' = \frac{\partial \psi}{\partial z} \quad w' = -\frac{\partial \psi}{\partial x}$
- perturbation $[\rho, p, \psi] = [\hat{\rho}(z), \hat{p}(z), \hat{\phi}(z)]e^{[ik(x-ct)]}$

$\rightarrow$

$$(U - c)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{N^2}{(U - c)} - U_{zz}\right\}\phi = 0$$

$$(U - c)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi + \left\{\frac{N^2}{(U - c)} - U_{zz}\right\}\phi = 0$$

Note that $c_{ph} = c - U$ is the phase velocity within the moving frame, and $\Omega = ck - Uk = \omega - Uk$ the Doppler shifted or intrinsic frequency.
It can be shown that for stability (see e.g. Drazin & Reid p327) :

$$Ri > 1/4$$

with Richardson number (also Ri) $= \frac{N^2}{(\partial U/\partial z)^2}$, N the Brunt Väisälä frequency)