RAYLEIGH-BÉNARD-, MARANGONI-, DOUBLE DIFFUSIVE-, CONVECTION

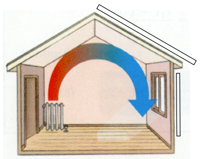
RAYLEIGH-BENARD CONVECTION

Convection

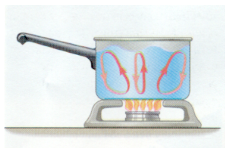
Heat transfer by diffusion
instability —>
Heat transport by convection

Forced Convection (pump, fan, suction device)
or
Natural convection (buoyancy force)

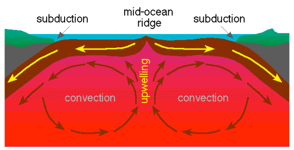
Solar panels



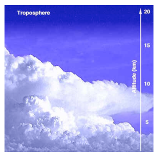
Buildings & Environment



industry



mantel of the Earth, mantel of the Sun...



Atmospheres

BENARD'S EXPERIMENT

T_1

T_0

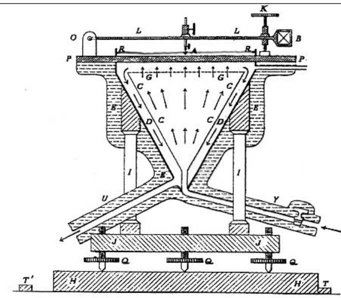


FIG. 10. The original apparatus of Bénard.

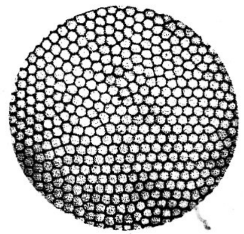


FIG. 1. Bénard cells in spermaceti. A reproduction of one of Bénard's original photographs.

HEAT CONVECTION

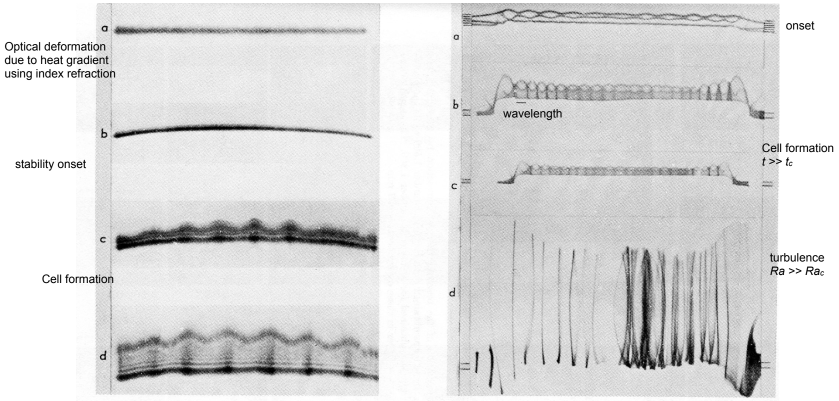
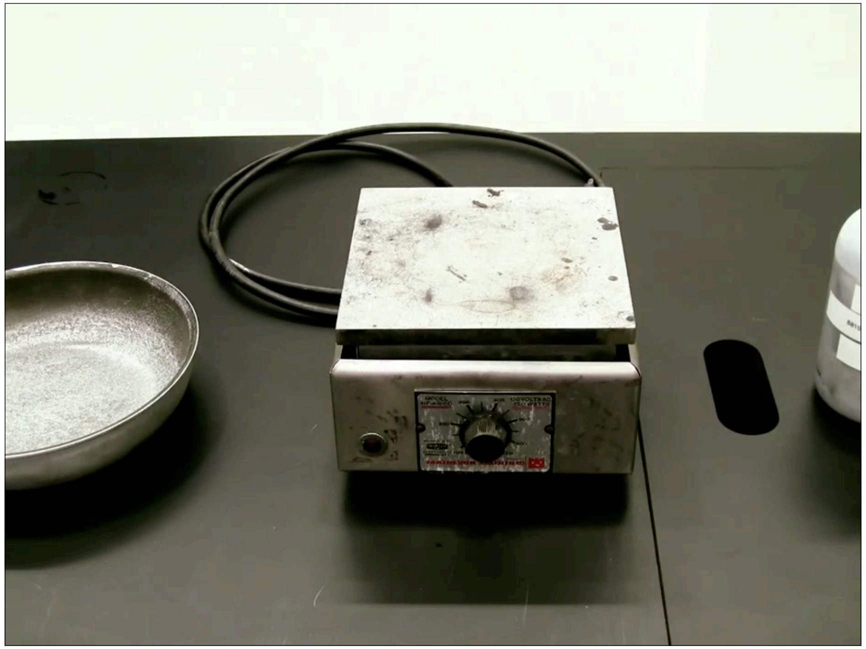


FIG. 15. Visualization of the onset of thermal convection by Schmidt and Milverton by the schlieren method (*Proc. Roy. Soc.* (*London*) A, **152**, 586 (1935)).

FIG. 16. Visualization of the onset of thermal convection by an optical arrangement by Schmidt and Saunders (*Proc. Roy. Soc.* (*London*) A, **165**, 216 (1938)). The data to which the illustrations refer are given below:

(*a*) $d = 1{\cdot}1$ cm, $R = 12{,}000$, $\Delta T = 0{\cdot}55°$ C
(*b*) $d = 0{\cdot}5$ cm, $R = 3{,}500$, $\Delta T = 1{\cdot}7°$ C
(*c*) $d = 0{\cdot}5$ cm, $R = 3{,}500$, $\Delta T = 1{\cdot}7°$ C
(*d*) $d = 1{\cdot}1$ cm, $R = 130{,}000$, $\Delta T = 4{\cdot}0°$ C

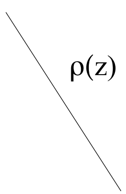


HEAT CONVECTION

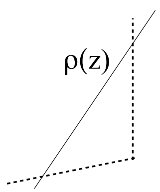
Driven by density gradients due to gradients in concentration, temperature or composition

STATIC STABILITY u=0

$$\rho(z) = \rho_0(1+\beta z) \qquad \beta = \alpha\frac{\Delta T}{h}$$



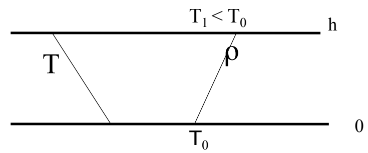
Stable



Statically unstable

parameters	g	ν	κ	β
Dimensions	LT^{-2}	L^2T^{-1}	L^2T^{-}	L^{-1}

RAYLEIGH BENARD CONVECTION



EQUATIONS With u=(u,v,w) and g=(0,0,-g)

Momentum $\rho\left(\frac{\partial u}{\partial t} + u.\nabla u\right) = -\nabla p - \rho g\mathbf{k} + \mu\nabla^2 u$

Diffusion of heat $\frac{\partial T}{\partial t} + u.\nabla T = \kappa\nabla^2 T$

Continuity $\nabla.u = 0$

Density $\rho = \rho(T) \qquad \frac{\rho(T)-\rho_0}{\rho_0} = -\alpha(T - T_0)$

STEADY SOLUTION

U=0

Hydrostatic balance

$\frac{dp}{dz} = -\rho(z)g$

$\frac{\rho(z)-\rho_0}{\rho_0} = -\frac{\alpha(T_0 - T_1)}{h}z$

Boussinesq approximation $\Delta\rho \approx 1\%$,

$(\rho - \rho_0)/\rho_0 = \alpha(T_0 - T) << 1$ only density variation in z considered

Note: Boussinesq approximation thus gives for density perturbations $\rho = \rho_0 + \rho'$ and related pressure perturbation $p = p_0 + p'$ with hydrostatic equilibrium :

$$-\frac{1}{\rho_0+\rho'}\frac{\partial p_0 + p'}{\partial z} - g = -\frac{1}{\rho_0(1+\rho'/\rho_0)}\frac{\partial p_0 + p'}{\partial z} - g$$

$$\approx -\frac{1}{\rho_0}\frac{\partial p_0 + p'}{\partial z} - \frac{\rho'}{\rho_0}\left(\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right) + ... - g = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0}g$$

Linearized perturbation equations around the basic state:

$u = (u',v',w')$; $T = T_0 + T'$ and with $\rho'/\rho_0 = \alpha T'$

into the equations and keep linear terms in the perturbation

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu\nabla^2 u' \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu\nabla^2 v' \qquad \nabla.\mathbf{u} = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\alpha T' + \nu\nabla^2 w' \qquad \frac{\partial T}{\partial t} - w\frac{T_0 - T_1}{h} = \kappa\nabla^2 T$$

X=x*H, t=t*H²/κ, u=u*H⁻¹κ, $p=p^*\rho_0\kappa^2/H^2$ T=T*ΔT

Substitution gives for the linearized equations:

$$\frac{\partial u}{\partial t} = -\nabla p + Ra\,Pr\,T\,\mathbf{k} + Pr\,\nabla^2 u$$

$$\frac{\partial T}{\partial t} - w = \nabla^2 T$$

$$\nabla.u = 0$$

Prandtl number

$$Pr = \frac{\nu}{\kappa} = \frac{\textit{diffusivity of momentum}}{\textit{diffusivity of heat}}$$

Rayleigh number

$$Ra = \frac{g\alpha\Delta T h^3}{\nu\kappa} = \frac{g\beta h^4}{\nu\kappa}$$

Pr = 7 for heat in water and ~O(100) for salt in water
0.7 in air and has very small values in liquid metals

Five equations, five unknown -->
eliminate the pressure with continuity to obtain a single equation by taking the curl of the velocity $\bar\omega = \nabla\times\bar u$ and using $\nabla.\bar u = 0$

$$\bar\omega = \nabla\times\bar u \rightarrow \frac{\partial\bar\omega}{\partial t} = Pr\nabla^2\bar\omega + Ra\,Pr\,\nabla T\bar k \qquad \text{(A)}$$

$$\nabla.\bar u = 0 \rightarrow \frac{\partial}{\partial t}\nabla^2\bar u = Ra\,Pr\left(\nabla^2 T\bar k - \nabla\frac{\partial T}{\partial z}\right) + Pr\nabla^4\bar u \longrightarrow$$

$$\longrightarrow \frac{\partial}{\partial t}\nabla^2 w = Ra\,Pr\nabla_h^2 T + Pr\nabla^4 w \quad \text{and} \quad \nabla_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{(B)}$$

$$\frac{\partial T}{\partial t} - w = \nabla^2 T \qquad \text{(C)}$$

Eliminate w or T from B, C to obtain a single equation in $\boldsymbol{T}$ or $\boldsymbol{w}$ (same eq.)

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + Pr\nabla^6 w - (1+Pr)\frac{\partial}{\partial t}\nabla^4 w = Ra\,Pr\,\nabla_h^2 w}$$

Note that from (A) we can solve the z-component

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \frac{\partial\omega_z}{\partial t} = Pr\nabla^2\omega_z$$

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + Pr\nabla^6 w - (1+Pr)\frac{\partial}{\partial t}\nabla^4 w = Ra\,Pr\,\nabla_h^2 w}$$

Normal modes $w(x,y,z) = Re\left[W(z)\exp i[(k_x x + k_y y) + st]\right]$

$$(\partial_z^2 - k^2 - s)(\partial_z^2 - k^2 - Pr^{-1}s)(\partial_z^2 - k^2)W + k^2 Ra W = 0$$

$$k^2 = k_x^2 + k_y^2$$

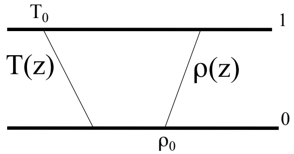
Boundary conditions --->

NOTE: FOR TAYLOR COUETTE FLOW, THE RELATION IS ≈ THE SAME

$$(DD_* - a^2 - \omega)(DD_* - a^2)\,u = -a^2 T\,g(x)\,v$$

$$(DD_* - a^2 - \omega)\,v = u$$

Boundary conditions



In all cases W=0 and T_z=0 at the boundaries z=0 and h (0 and 1 scaled)
the temperature is maintained at a constant at z=0 and h

*A) Bounding surfaces are rigid: **no slip** (zero tangential velocity) most realistic*

u=0, v=0 at z=0, 1. Since this is valid for all x and y, this is also valid for u_x=0 and v_y=0 so that

$$\nabla_h . u = 0 \rightarrow \frac{\partial w}{\partial z} = 0 \text{ and idem for vorticity } \omega_z = 0$$

*B) Bounding surfaces are free surfaces: **free slip** (zero tangential stress) (easiest!)*

$$\tau_{xz} = \tau_{yz} = 0 \;\rightarrow \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial v}{\partial z} = 0$$

$$\rightarrow \frac{\partial}{\partial z}\nabla . \bar{u} = 0 \rightarrow \frac{\partial^2 w}{\partial z^2} = 0 \text{ at } z = 0, 1$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = \frac{\partial \omega_z}{\partial z} = 0 \text{ at } z = 0, 1$$

$$\text{with equations } \frac{\partial^4 w}{\partial z^4} = 0 \text{ at } z = 0, 1$$

Stress free, perfectly conducting boundaries (Rayleigh 1916)

$$(\partial_z^2 - k^2 - s)(\partial_z^2 - k^2 - Pr^{-1}s)(\partial_z^2 - k^2)W + k^2 RaW = 0$$

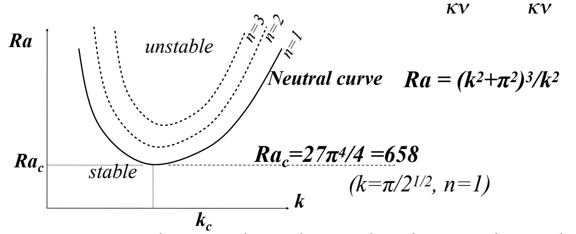
Marginal stability curve s=0 (Neutral stability curve)

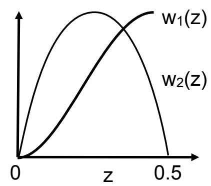
$$[(\partial_z^2 - k^2)^3 + Ra\, k^2]\, W = 0$$

Boundaries: $W=\partial_z^2 W=(\partial_z - k^2)^2 W=0 \; (z=0,1)$

Solution $W=\sin n\pi z$ with $(n^2\pi^2+k^2)^3=Ra\, k^2$

First unstable mode for n=1 $Ra = \frac{\alpha g \Delta T h^3}{\kappa\nu} = \frac{\beta g h^4}{\kappa\nu}$





Ra_{crit} does not depend on *Pr* but the growth rate does!

NOTES

- critical value O(10^3) depends on boundary conditions
  Realistic temp differences result in Ra>>Ra_c
  Ra increases rapidly with h: deeper layers produce higher Ra numbers !

at 20^0 $\alpha_{air} \approx 0.003 K^{-1}$
$\alpha_{water} \approx 0.0002\ K^{-1}$

$$Ra = \frac{\alpha g \Delta T h^3}{\kappa\nu} = \frac{\beta g h^4}{\kappa\nu}$$

- Lowest mode n=1 has the least vertical structure. Vertical velocity has only one sign but changes sign for higher modes.

  (correlation between vertical motion and buoyant fluid results from the momentum generated by the buoyancy forces.)

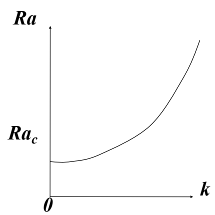
- linear analyses determines only the horizontal scale and not the structure !

Perfectly conducting boundaries

Free and one rigid boundary (Benard problem) $Ra_c \approx 1101$ at $k_c = 2.68$

Two rigid boundaries $Ra_c \approx 1708$ at $k_c = 3.12$

With fixed heat fluxes T_z=*constant* at boundaries (different T-curve)



Two stress free boundaries Ra_c= 120
Two rigid boundaries Ra_c= 720

Flows related to Rayleigh Benard !

TAB. 10.1 – *Tableau comparatif des paramètres caractéristiques des instabilités de Rayleigh–Bénard, Taylor–Couette et Bénard–Marangoni. Notons que, pour les deux premières, la correspondance s'étend jusqu'à donner des valeurs numériques très voisines pour les seuils.*

$(\eta = \mu)$	Instabilité de Rayleigh–Bénard	Instabilité de Taylor–Couette	Instabilité de Bénard–Marangoni
force de freinage visqueuse	$F_{\mathrm{visc}} = \eta\, v_c\, a$	$F_{\mathrm{visc}} = \eta\, v_c\, a$	$F_{\mathrm{visc}} = \eta\, v_c\, a$
force motrice	$F_{\text{poussée d'Archimède}}$ $\rho_0\, \alpha\, g \frac{a^5}{\kappa} \frac{\Delta T}{a} v_c$	$F_{\text{force centrifuge}}$ $\rho_0\, a^2 \frac{\Omega^2 R\, a^5}{a\ \ \nu} v_c$	$F_{\text{tension superficielle}}$ $\frac{a^3}{\kappa} \frac{d\gamma}{dT} \frac{\Delta T}{a} v_c$
temps caractéristique de relaxation de la perturbation avec le fluide environnant	$\frac{a^2}{\kappa}$	$\frac{a^2}{\nu}$	$\frac{a^2}{\kappa}$
paramètre caractéristique de l'instabilité	$Ra = \frac{\alpha\, \Delta T\, g\, a^3}{\nu\, \kappa}$	$Ta = \frac{\Omega^2 R\, a^3}{\nu^2}$	$Ma = \frac{\frac{d\gamma}{dT} \Delta T\, a}{\eta\, \kappa}$
valeurs critiques d'apparition des instabilités	$Ra_c = 1\,708$ $k_c = \frac{3,11}{a}$	$Ta_c = 1\,712$ $k_c = \frac{3,11}{a}$	$Ma_c = 80$

Guyon, Hulin et Petit, 1982

How do we measure the transition to convection ?

Heat Flux : The Nusselt number

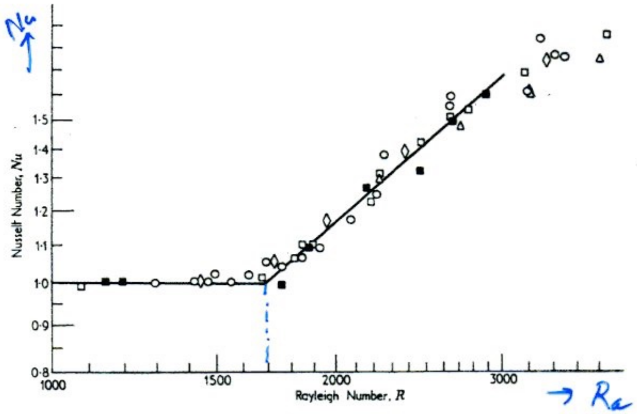
Heat transfer rate F=λ(ΔT)/H, λ is heat transfer coefficient

$$Nu = \frac{F H}{\lambda\, \Delta T}$$

$$Nu = \frac{\textit{total heat flux}}{\textit{conducted heat flux}}$$

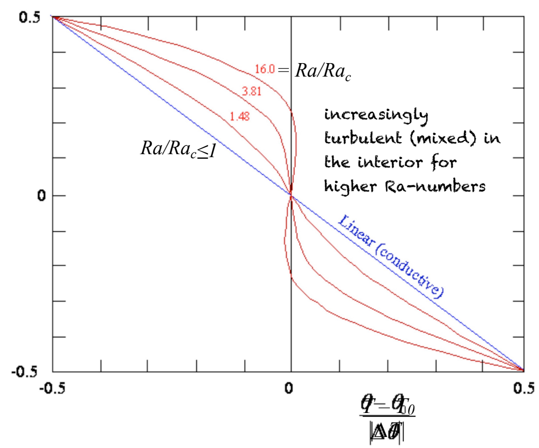
In unsteady flows the Nusselt number is averaged in time

no flow Nu=1
flow Nu>1



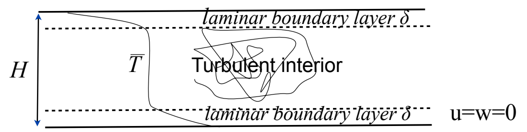
onset of Rayleigh Benard convection

Boundary layer structure



For larger Ra the boundary layer thickness decreases
Interior ≈ homogenous

Turbulence convection

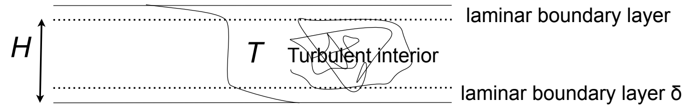


heat flux in each boundary layer δ is $F_\delta = \lambda T_\delta / \delta$

the Nusselt number for the boundary layer is then $\left(Nu = \frac{F_\delta}{\lambda \Delta T / H} \right)$

$$Nu = \frac{T_\delta H}{\Delta T \delta}$$

From observations $T_\delta \approx \Delta T/2$ so that Nu $\approx H/2\delta$



the boundary layer thickness grows with $\delta \sim (\kappa t)^{1/2}$ until it becomes unstable and releases its heat (see*)

for the boundary layer δ $Ra_\delta = \frac{\alpha g (\Delta T/2) \delta^3}{\kappa \nu} = Ra_{\delta\ crit.} = 1708.$

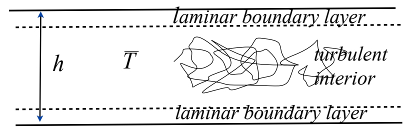
$$\Rightarrow \quad \frac{\delta_c}{H} = \left(\frac{2 Ra_{\delta\ crit}}{Ra_H} \right)^{1/3}$$

the scaling of the Nusselt number was Nu $\approx H/2\delta$:

$$Nu = \frac{H}{2\delta} = \frac{1}{2} \left(\frac{1}{2} \frac{Ra}{Ra_c} \right)^{1/3} \approx 0.077 Ra^{1/3}$$

(*note: there are other scalings, see e.g. Lohse & Xia Ann. Rev Fluid Mech 2009, Ahlers et al (2009))

Turbulence convection



HEAT FLUX => $F_H = \frac{k \Delta T}{h} Nu$

in the interior $Nu \sim Ra^{1/3}$

Interior is turbulent and uniform : $\boldsymbol{F_H}$ independent of $\boldsymbol{h}$,

Then we have for a fixed Prandtl number $\boldsymbol{P}$

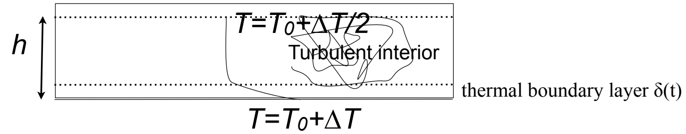
$$Nu = f(Ra, P) \propto Ra^{1/3} = \left(\frac{\alpha g \Delta T h^3}{\kappa \nu} \right)^{1/3}$$

and for the turbulent heat flux:

$$F_H = c(P) \left(\frac{\alpha g k^3}{\nu \kappa} \right)^{1/3} \Delta T^{4/3} = C\ \Delta T^{4/3}$$

Also called *4/3 law*. (Howard, Appl Mech.1964)

*Plumes released from the boundary



Howard 1966

The thermal BL grows by thermal diffusion

$$T = T_0 + \frac{\Delta T}{2} \left(1 + \operatorname{erfc} \frac{z}{2\sqrt{\kappa t}} \right) \qquad \delta = \sqrt{\pi \kappa t}$$

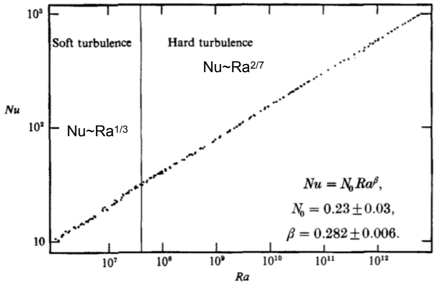
this goes on until it is unstable and breaks down at t_c *(i.e. at Ra_c)* and δ breaks down to zero:

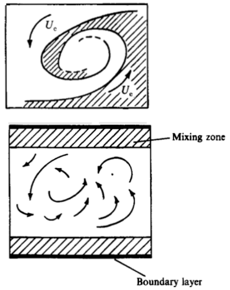
$$\frac{\alpha g \Delta T \delta_c^3}{2\kappa\nu} = Ra_c \quad \Rightarrow \quad \delta_c = \left(\frac{2\kappa\nu}{\alpha g \Delta T} \right)^{1/3} Ra_c^{1/3}$$

the time cycle of the plume formation can be calculated as

$$t = \frac{h^2}{\pi\kappa} \left(\frac{Ra_c}{Ra} \right)^{2/3}$$

Castaing et al, J. Fluid Mech. 1989: boundary layer relation with turbulent interior





possible cartoons for the flow pattern in the cell centre.
3 regions: BL, mixing zone, turbulent interior

recent research:

- **Geometry effects:** geometry modifies temperature and vertical velocity fluctuations at the cell center and their Rayleigh-number dependence; interior turbulent fluctuations are nonuniversal in low-aspect ratio convection (Daya &Ecke 2001 Phys Rev Lett; for tilt of container Chilla et al 2004).
- A possible reason are different thicknesses of the kinetic boundary layers at the sidewalls and at the top/bottom plates (Lohse & Xia Ann. Rev Fluid Mech 2009, Ahlers et al (2009), Rev Mod Phys 81:503) ...

Turbulence convection
general comments

These laws are also known as Malkus, Kraichnan, Howard, Ginsburg-Landau (GL) model. The general law is written as

$$Nu = C\, Ra^{\beta}\, \mathrm{Pr}^{\gamma}$$

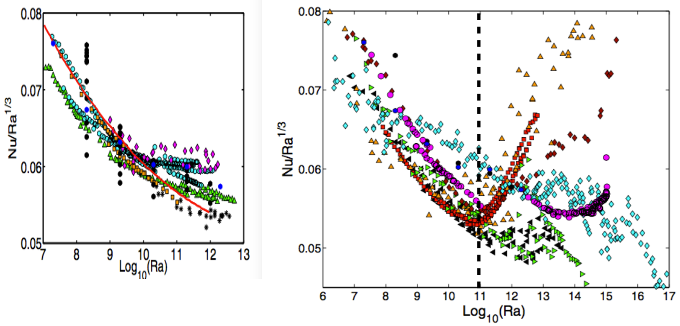
allows to represent all models for infinite horizontal dimensions. For a finite horizontal size, D, the constant C=C(Γ)depends on the aspect ratio Γ=D/H.

Malkus (1954) proposed β= 1/3 and γ= 0 supposing a heat flux that is independent with height.

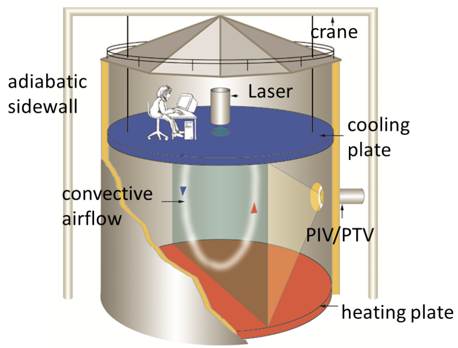
Kraichnan's law, β= 1/2, and boundary layers can become unstable.

The GL theory proposes a phenomenological law with a linear combination of expressions like above and some adjustable parameters that easily allow to fit the data. In the fully turbulent regime, for $10^6 < Ra < 10^{11}$, all experiments agree within an error of 10%

For Ra > 10^{11} there is a clear divergence up to a even a factor 2 in Nusselt number

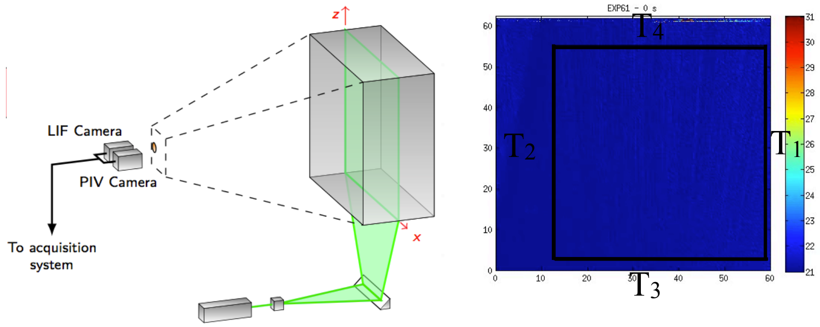


Present state of the art of heat transfer power law for very high Ra number
Chilla & Schumacher 2012



Some RB Convection setups
Ilmenau, Germany (Eberhard Bodensatz)
ENS Lyon (Francesca Chilla)
Inst Neel Grenoble (Philippe Neel) Cryogenic
Eindhoven Pays Bas (rotating convection, Rudie Kunnen)
NLNA US, (Bob Ecke)
Santa Barabara, (US Günter Ahlers)
HongKong China (Ke-Qing Xia)
….

Expériences de laboratoire LEGI



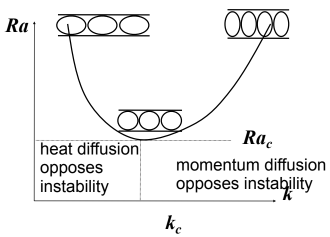
LEGI Grenoble

CONVECTION

PATTERN FORMATION

Dependence on Prandtl number

$$n = 1 \quad s = \frac{(h^2 + \pi^2)(Ra - Ra_c)}{(1 + Pr^{-1})Ra_c}$$



Ra_c is the critical Rayleigh number, $Ra > Ra_c$

The Prandtl number does affect the growth rate and convection type instability (even at large *Ra* and large *Pr*)

For **large *Pr-number*** ν/κ>>1 (e.g. salt and momentum diffusion)
Faster momentum diffusion!!
large Ra number (ΔT large):
Large (broad) convection cells
temperature anomalies in *narrow regions*

For **small *Pr-number*** ν/κ<1 (liquid metals etc.)
Narrow convection cells (slower momentum diffusion, faster heat diffusion)
temperature anomalies broaden into large regions.

Rayleigh-Benard Convection

high aspect ratio

pattern control

without defects
high Pr — low Pr
rolls — rolls rolls
skew -varicose (larger rolls) — *cross-rolls*
oscillating rolls — *oscillating rectangles*

with defects
low Pr — high Pr
rolls with imperfection
dislocation movement →
← *localised oscillations*

loss of periodicity
spatial structure; temporal randomness
loss of spatial structure
Ra>10⁴ turbulent convection (earlier for small Pr<1)

low aspect ratio

Route to chaos:

period doubling
Quasi periodicity
intermittency

Ra ↓

illustrations of convection patterns →

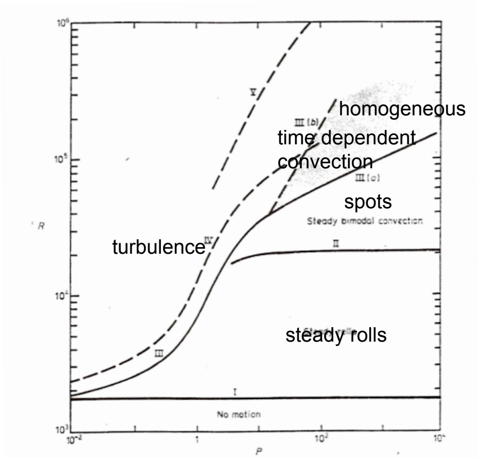
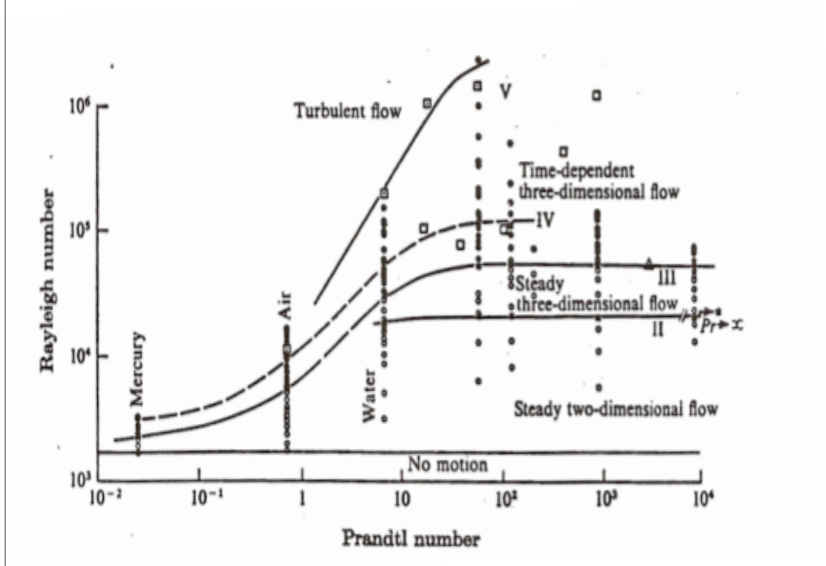


Figure 5. Transitions in convection as a function of Rayleigh and Prandtl numbers after Krishnamurti (1973) and others. The curves indicate the onset of steady rolls (I), three-dimensional convection pattern (II), time-dependent convection (III) in isolated spots (III(*a*)) and uniformly throughout the layer (III(*b*)), and turbulent convection (IV).



Cell patterns

Disturbances are characterized by a particular wave number, but the convection pattern is unspecified, because the wave vector k^2 can be resolved into k_x and k_y in infinitely many ways.

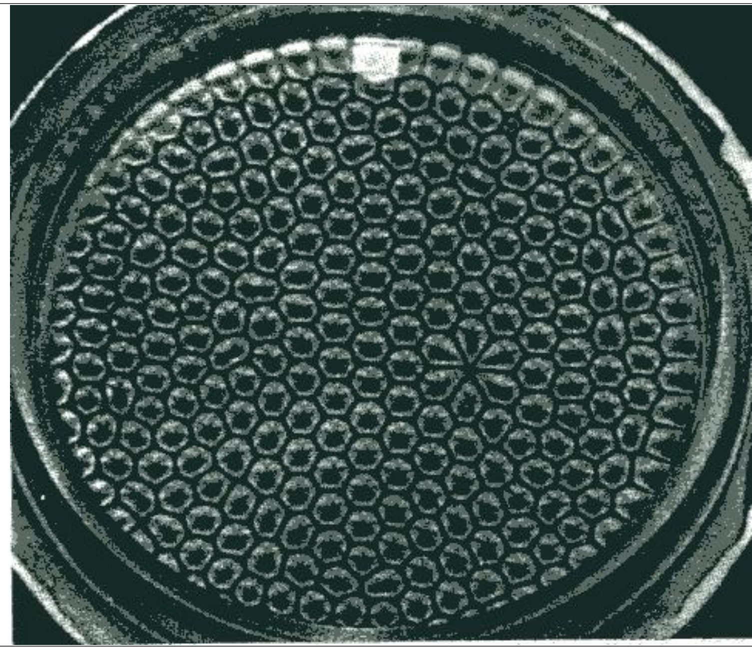
One has to consider a certain symmetry in advance in θ (x,y,z,t) or w(x,y,z,t)
For instance for

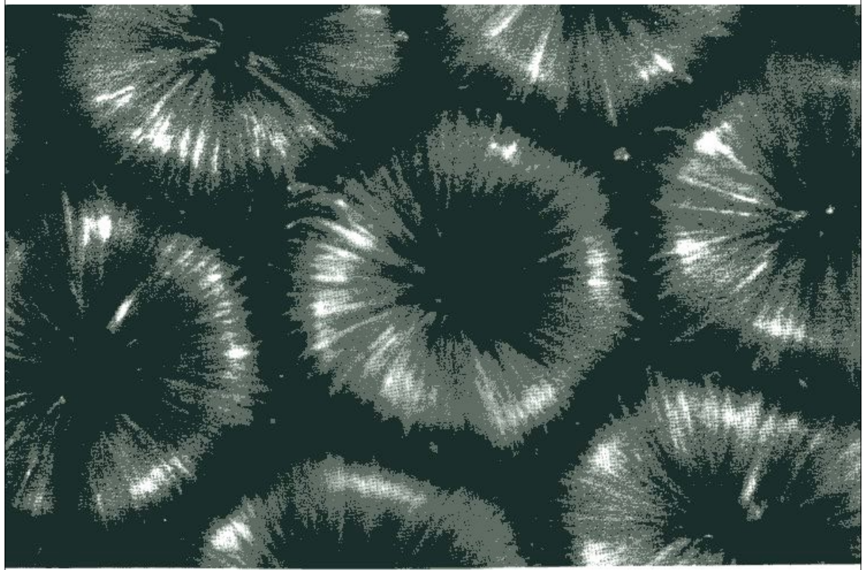
rolls: $\theta = \theta(z) \cos k \cdot x$ $\quad k = (k_x^2 + k_y^2)^{1/2}$

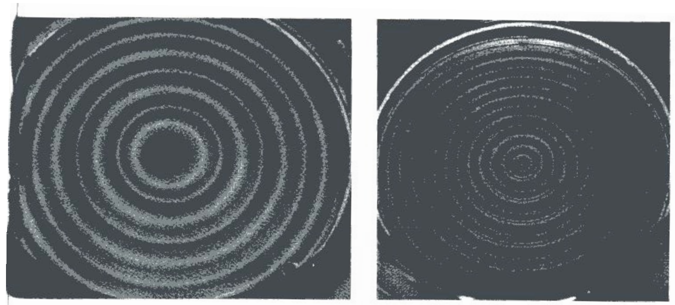
Rentangular & square cells: $\theta = \theta(z) \cos k_x x \cos k_y y$

Hexagonal cells: $\theta = \theta(z)\,[2\cos k\, x/\sqrt{3}\, \cos k\, y/\sqrt{3} + \cos 2/3\, k\, y\,]$
$k = (k_x^2 + k_y^2)^{1/2} - 2\pi/L$
and L the size of the hexagon

e.g Chandrasekhar 1961 p 43-52.

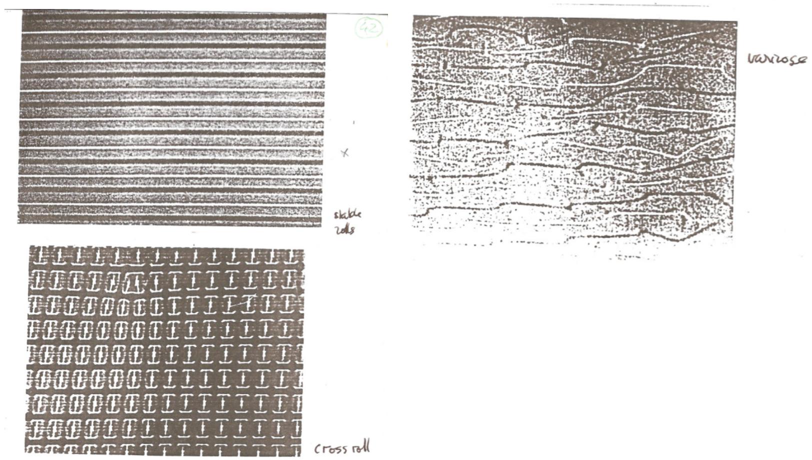


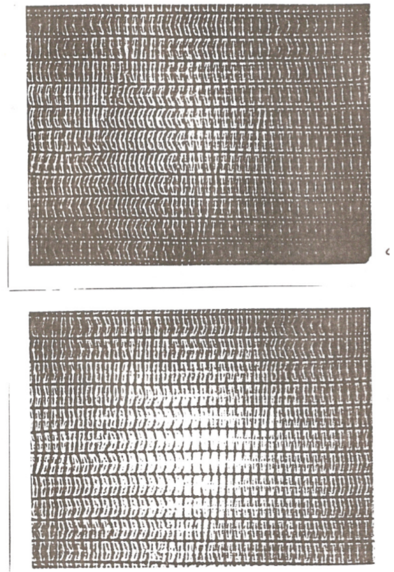




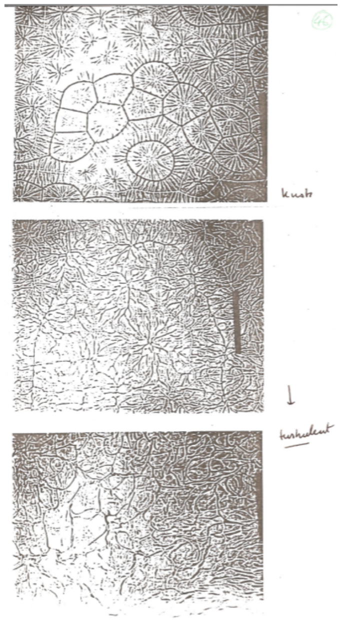
140. Circular buoyancy-driven convection cells. Silicone oil containing aluminum powder is covered by a uniformly cooled glass plate, which eliminates surface-tension effects. The circular boundary induces circular rolls. In the left photograph the copper bottom is uniformly heated at 2.9 times the critical Rayleigh number, giving regular rolls. At the right, the bottom is hotter at the rim than at the center. This induces an overall circulation which, superposed on regular circular rolls, produces alternately larger and smaller rolls. Koschmieder 1974, 1966

Flow structures with increasing Ra-number



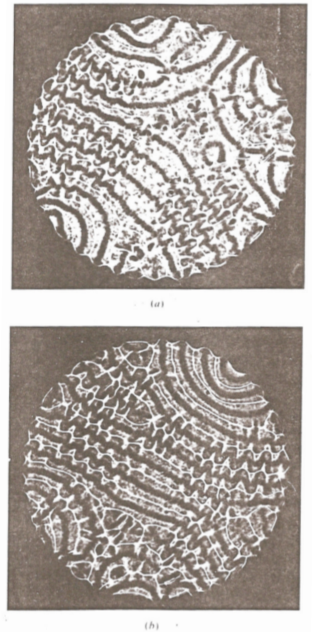


oscillatory



« natural »
Convection
No structure
In the forcing

Osillatory
instability



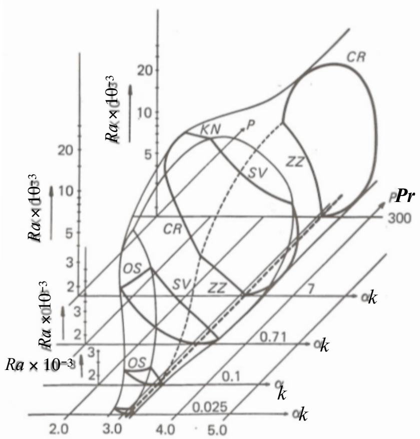
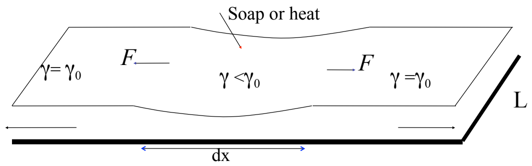


Fig. 5.12. Stability domain of convection rolls in the $R-P-\alpha$ space. The thick curves represent computed stability boundaries for the oscillatory (OS), the skewed varicose (SV), the cross-roll (CR), the knot (KN), and the zigzag (ZZ) instabilities after [5.38, 112, 113]. For $P=300$, the results for $P=\infty$ [5.45] have been used   Busse 1981 Springer ed. Swinney & Gollub 1981

MARANGONI OR RAYLEIGH -BENARD CONVECTION ?

MARANGONI EFFECT

Some solubles (soap) or temperature gradients create surface tension.



This results in a tangential stress and deformation of the liquid layer for small temperature gradients, $\gamma(T) = \gamma(T_0)(1-b(T-T_0))$

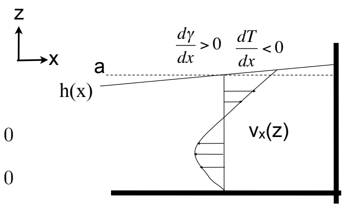
surface tension $\quad \frac{d\gamma}{dx} = \frac{d\gamma}{dT}\frac{dT}{dx} = -b\gamma(T_0)\left(\frac{dT}{dx}\right)$

at the free surface : $\quad \sigma^{\gamma}_{xz} = \frac{dF}{Ldx} = \frac{d\gamma}{dx} = -b\gamma(T_0)\left(\frac{dT}{dx}\right)$

viscous stress: $\quad \sigma^{\nu}_{xz} = -\mu \frac{\partial v_x}{\partial z}$

$$\boxed{\sigma^{\gamma}_{xz} + \sigma^{\nu}_{xz} = 0}$$

Near a wall



$$\int_0^a v_x(z)dz = 0$$

$$v_z(0) = 0$$

See Guyon, Hulin & Petit p 250

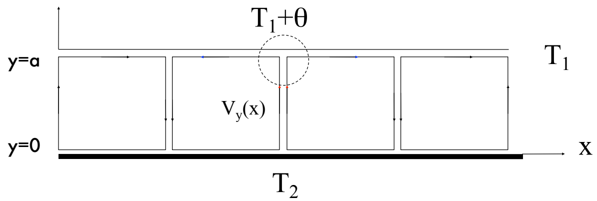
for the pressure we have: $p = p_{atm} + \rho g(a-z)$
Since the pression does not depend on x and the fluid is at rest, NS gives near the wall:

$$\eta \frac{\partial^2 v_x}{\partial z^2} = \rho g \frac{dh}{dx}$$

with the balance above and boundary conditions we obtain then:

$$v_x(z) = -\frac{\rho g}{\eta}\frac{dh}{dx}\left[\frac{z^2}{2} - \frac{az}{3}\right] \qquad \frac{dh}{dx} = -\frac{3}{2}\frac{b\gamma(T_0)}{\rho g a}\frac{dT}{dx}$$

BENARD-MARANGONI CONVECTION



Motion driven by surface tension, heat diffuses

By continuity hot fluid aliments the process

viscosity and diffusion oppose the convective motion as in Rayleigh Benard convection

Marangoni number $\quad Ma = \frac{b\gamma \Delta T a}{\mu\kappa}$ and $b = \frac{1}{\gamma}\frac{\partial\gamma}{\partial T}$

Its derivation is similar to that of the Rayleigh number Ra

The ratio is $\frac{Ra}{Ma} = \frac{\alpha\rho g a^2}{d\gamma/dT} = \frac{|\delta\rho| g a^2}{|\delta\gamma|}$ (same δT for δρ and δγ)

Equations are given by

$$\nabla \cdot \boldsymbol{v} = 0,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v},$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \kappa \nabla^2 T.$$

Boundary conditions

bottom $\quad \boldsymbol{v} = 0, \quad \partial_z T - Bi^b\, T = \text{constant},$

top $\quad w = 0, \quad \partial_x \sigma = \mu \partial_z u, \quad \partial_y \sigma = \mu \partial_z v, \quad \partial_z T + Bi^t\, T = \text{constant}$

heat transfer at the boundaries (with *Bi* the Biot number, *Bi^t* indicating the heat transfer at the *top (t) or bottom (b)* of the boundary layer, λ is the thermal conductivity)

$$Bi^t \approx \frac{\lambda_{air}}{\lambda}\frac{k}{\tanh k d_{air}}$$

For discussions of *Bi* see Bragard & Velarde JFM 1998

Linear stability analyses:

$$T = T_{ref} + T'$$

perturbation $\quad T' = T(z)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$

$$\longrightarrow$$

$$\left(\mathrm{D}^2 - k^2\right)^2 W(z) = 0,$$

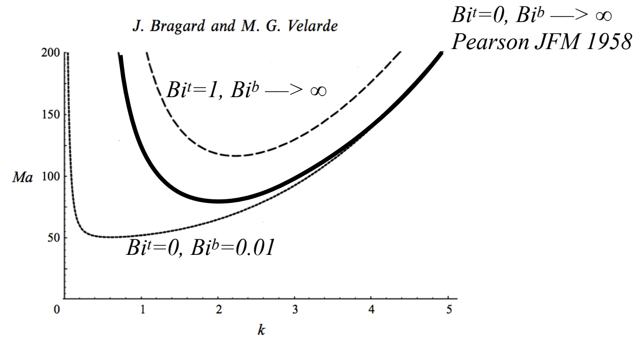
$$\left(\mathrm{D}^2 - k^2\right) T(z) + W(z) = 0$$

$z = 0 \qquad W = 0; \quad \mathrm{D}W = 0; \quad \mathrm{D}T - Bi^b T = 0$

$z = 1 \qquad W = 0; \quad \mathrm{D}^2 W + k^2 Ma\, T = 0; \quad \mathrm{D}T + Bi^t T = 0$

The Marangoni number is $\quad Ma = \dfrac{-(\partial\sigma/\partial T)\Delta T h}{\mu\kappa}$

see Bragard & Velarde JFM 1998 for further details



Amplitude equations etc. to consider the structures above the critical Ma number (see Bragard & Velarde 1998)

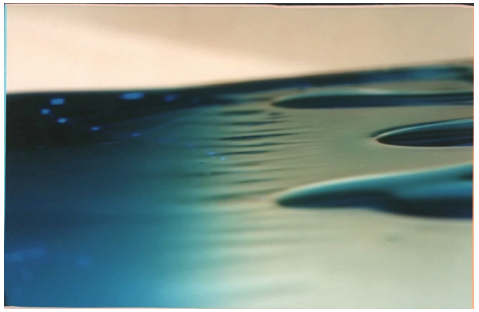
Effect was first observed by James Thompson 1855 in a glass of wine

$\gamma_{alcohol} < \gamma_{water}$

different alcohol concentrations

liquid flows away from regions with lower alcohol concentration (=higher tension)

What happens ?
Explain




Skotheim, J.M, and Bush, J.W.M., 2000. Evaporatively-driven convection in a draining soap film, *Gallery of Fluid Motion, Physics of Fluids* , **12** (9)

Hosoi, A.E. and Bush, J.W.M., 2000. Evaporative instabilities in climbing films, *J. Fluid Mech.* , **442,** 217-229

example in paints

Flows related to Rayleigh Benard !

TAB. 10.1 – *Tableau comparatif des paramètres caractéristiques des instabilités de Rayleigh–Bénard, Taylor–Couette et Bénard–Marangoni. Notons que, pour les deux premières, la correspondance s'étend jusqu'à donner des valeurs numériques très voisines pour les seuils.*

$(\eta = \mu)$	Instabilité de Rayleigh–Bénard	Instabilité de Taylor–Couette	Instabilité de Bénard–Marangoni
force de freinage visqueuse	$F_{\text{visc}} = \eta\, v_c\, a$	$F_{\text{visc}} = \eta\, v_c\, a$	$F_{\text{visc}} = \eta\, v_c\, a$
force motrice	$F_{\text{poussée d'Archimède}}$ $\rho_0\, \alpha\, g \frac{a^5}{\kappa} \frac{\Delta T}{a} v_c$	$F_{\text{force centrifuge}}$ $\rho_0\, a^2 \frac{\Omega^2 R\, a^5}{a \quad \nu} v_c$	$F_{\text{tension superficielle}}$ $\frac{a^3}{\kappa} \frac{d\gamma}{dT} \frac{\Delta T}{a} v_c$
temps caractéristique de relaxation de la perturbation avec le fluide environnant	$\frac{a^2}{\kappa}$	$\frac{a^2}{\nu}$	$\frac{a^2}{\kappa}$
paramètre caractéristique de l'instablité	$Ra = \frac{\alpha\, \Delta T\, g\, a^3}{\nu\, \kappa}$	$Ta = \frac{\Omega^2 R\, a^3}{\nu^2}$	$Ma = \frac{\frac{d\gamma}{dT} \Delta T\, a}{\eta\, \kappa}$
valeurs critiques d'apparition des instabilités	$Ra_c = 1\,708$ $k_c = \frac{3,11}{a}$	$Ta_c = 1\,712$ $k_c = \frac{3,11}{a}$	$Ma_c = 80$

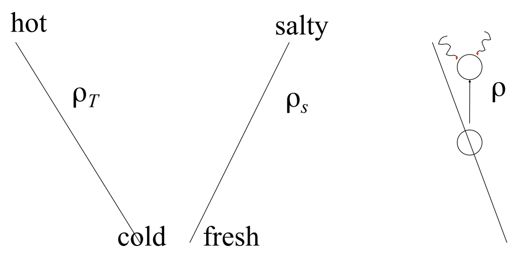
Guyon, Hulin et Petit, 1982

DOUBLE DIFFUSIVE CONVECTION

DOUBLE DIFFUSIVE CONVECTION

Stable stratification and *unstable*…
Two or more stratifying agents compose a stable density stratification but diffuse at different rates.

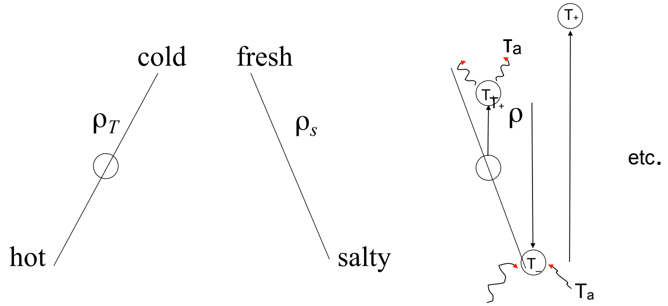
Opposing effects: heat diffusion and dissipation due to viscous effects



$\kappa_s/\kappa_T = 10^{-2}$ for salt and heat (1/3 for sugar and salt)

Consider a blobs displacement (up and down) and see whether it is stable

OSCILLATORY DIFFUSIVE CONVECTION



Consider a blobs displacement and see whether it is stable….

Over-stable oscillation = oscillation but unstable !
i.e. $w \sim e^{i\omega t+ikx}$ with $\omega=\zeta+i\xi$ $\zeta \neq 0$ and $\xi \neq 0$!

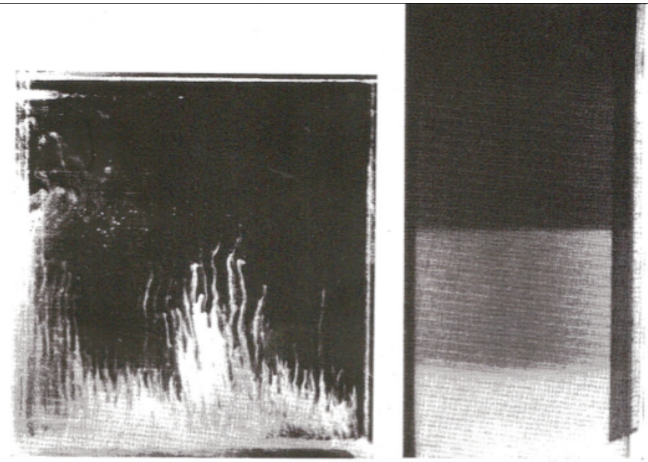


Fig. 8.8. (*Left.*) Vertical cross-section of salt fingers, marked by fluorescein dye added to the upward moving fingers, and lit with a thin sheet of light perpendicular to the viewing direction.

Fig. 8.9. (*Right.*) The formation of convecting layers from a smooth stable gradient of salt, driven by a flux of sugar originating in the dark dyed layer at the top. (From Stern and Turner, *Deep-Sea Res.* **16**, 497–511. © Pergamon Press Ltd, 1969.)

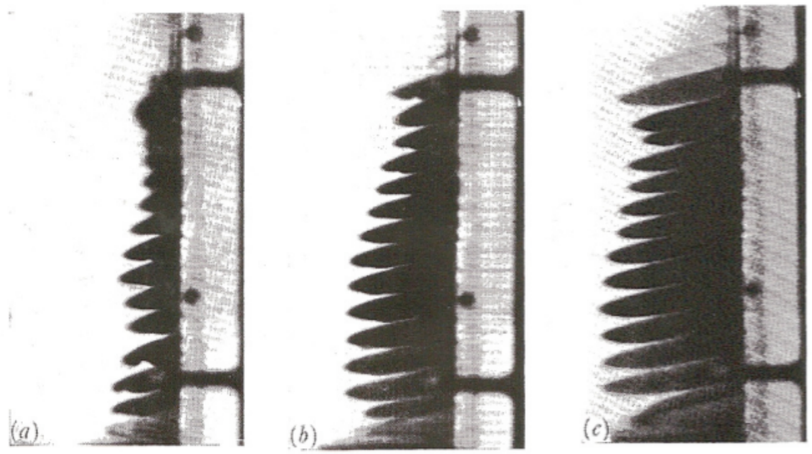
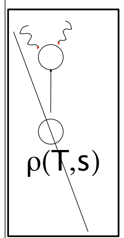
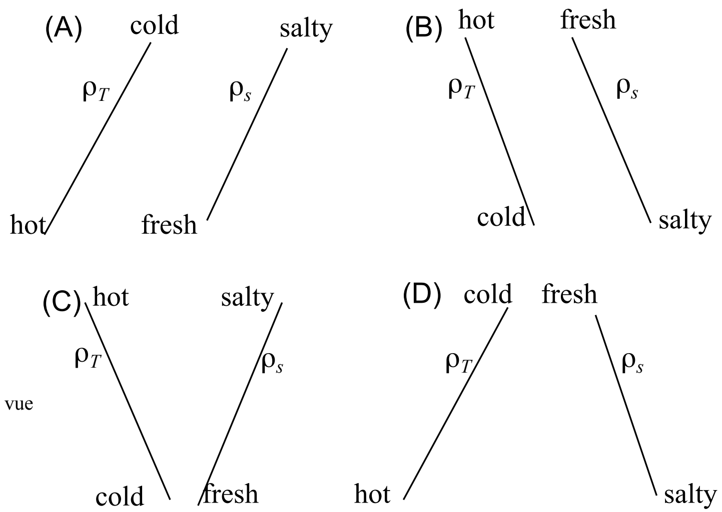


Fig. 8.10. The development of layers in a stratified salt solution subject to heating through a vertical side wall. The photographs were taken (*a*) 19.5 min, (*b*) 24 min and (*c*) 28.25 min after heating began. (From Thorpe, Hutt and Soulsby 1969.)

Consider different cases of diffusion of salt &heat $\rho(S,T) = \rho_m(1 - \alpha T + \beta S)$

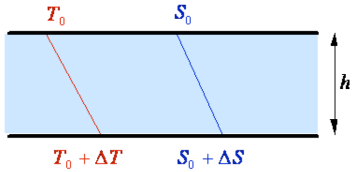




C et D deja vue

Layer formation ...

EXTENDED RAYLEIGH-BENARD PROBLEM



After linearization the equations for salt and heat are

$$\frac{\partial S}{\partial t} + \frac{\partial \psi}{\partial x} = \tau \nabla^2 S \qquad \tau = \frac{\kappa_s}{\kappa_T} \qquad \frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial x} = \nabla^2 T$$

and the vorticity equation is

$$P^{-1}\nabla \psi_t = -R_T \frac{\partial T}{\partial x} + R_s \frac{\partial S}{\partial x} + \nabla^4 \psi$$

$$R_s = \frac{\beta g \Delta S h^3}{\kappa_S \nu} \qquad R_T = \frac{\alpha g \Delta T h^3}{\kappa_T \nu} \qquad P = \frac{\nu}{\kappa_T}$$

linear stability anal. Schmitt Ann. Rev Fluid Mech 1994;
Baines & Gill 1969 J. Fluid Mech

More precisely (following Baines and Ghil 1969)
Linear stability analyses:

$$\rho = \rho_m(1 - \alpha T^* + \beta S^*)$$

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T^*}\right)_{S,p^*}, \quad \beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial S^*}\right)_{T,p^*}$$

Boussinesq perturbation equations are

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi = -R\frac{\partial T}{\partial x} + R_s\frac{\partial S}{\partial x}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) T = -\frac{\partial \psi}{\partial x},$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right) S = -\frac{\partial \psi}{\partial x}, \qquad \mathbf{u} = (u, w) = \left(-\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x}\right)$$

$$R = \frac{g\alpha \Delta T d^3}{\nu \kappa_T}, \quad R_s = \frac{g\beta \Delta S d^3}{\nu \kappa_T}$$

$$\mathbf{u}^* = \frac{\kappa_T}{d}\mathbf{u}, \quad t^* = \frac{d^2}{\kappa_T}t, \quad x^* = dx, \quad y^* = dy, \quad z^* = dz,$$

$$T^* = \Delta T \,.\, T, \quad S^* = \Delta S \,.\, S, \quad \psi^* = \kappa_T \psi.$$

Boundary conditions at z=0,1 are:

$$\psi = \frac{\partial^2 \psi}{dz^2} = T = S = 0$$

Solutions are of the form

$$\psi \sim e^{pt} \sin \pi \alpha x . \sin n\pi z,$$
$$T, S \sim e^{pt} \cos \pi \alpha x . \sin n\pi z$$

The dispersion relation for $p=k^2 q$ is then (with $k^2=\pi^2(\alpha^2+n^2)$)

$$F(q) \equiv q^3 + (\sigma + 1 + \tau)q^2 + [\sigma + \sigma\tau + \tau - \sigma(R' - R_s')]q + \sigma(\tau + R_s' - \tau R') = 0$$

$$R' = \pi^2\alpha^2 R/k^6, \quad R_s' = \pi^2\alpha^2 R_s/k^6$$

From this relation the different instability regimes as a function of R' and R'_s are obtained (see Baines & Ghil 1969).

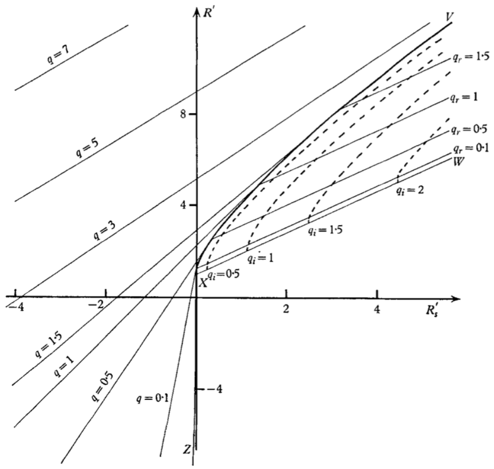
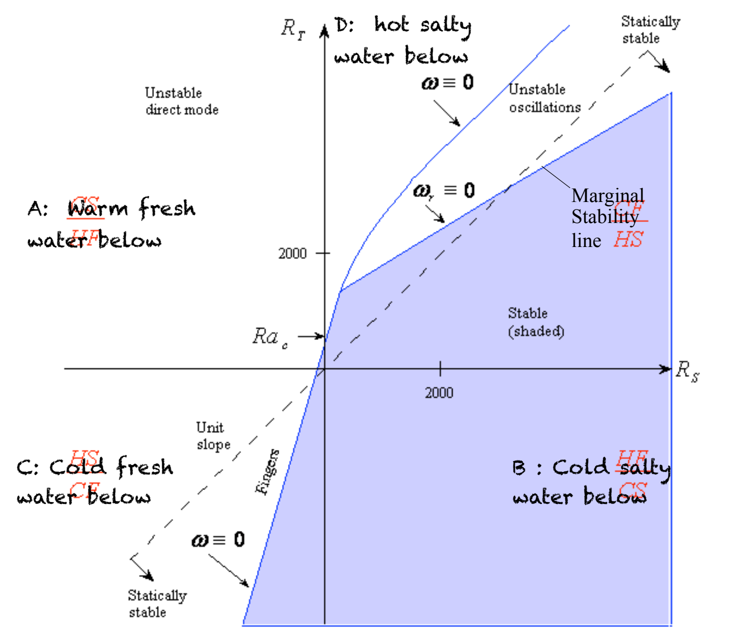
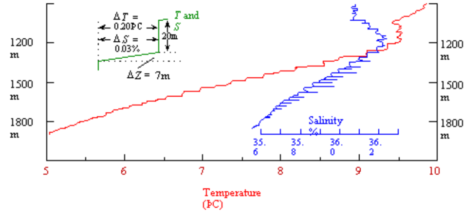


FIGURE 2. Lines of constant growth rate and constant frequency (shown dashed) in the R', R_s' plane.



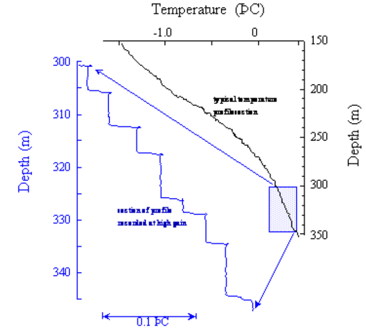
Examples 1: The ocean

3.20



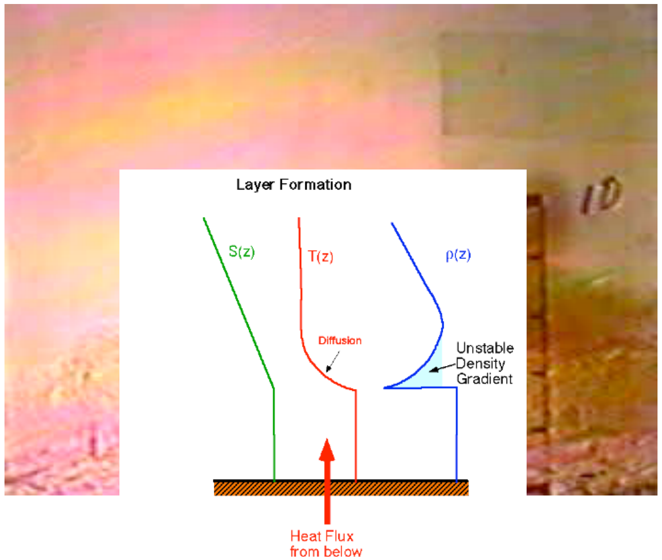
A series of layers in T and S observed under the Mediterranean outflow into the Atlantic, and attributed to the 'finger' mechanism. The sharp spikes to the right of the salinity trace are instrumental and can be ignored. The inset summarizes the mean properties of the layers and interfaces.

(Tait & Howe, 1971)



Temperature profile under Arctic Ice Island T3, showing steps formed by the double-diffusive mechanism.

(Neal, Neshyba & Denner, 1969)

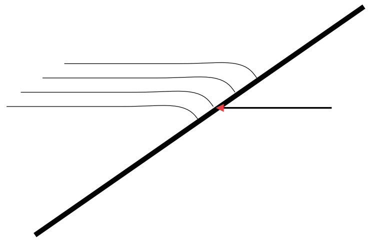


heated bottom below a salt gradient

Examples 2: Coffee

Remark

Inclined isolated slopes



Ficks law at the wall is: $\partial\rho/\partial n=0$

SUMMARY

-BENARD MARANGONI CONVECTION

-RAYLEIGH BENARD CONVECTION

-HEAT TRANSPORT

-DOUBLE DIFFUSIVE CONVECTION