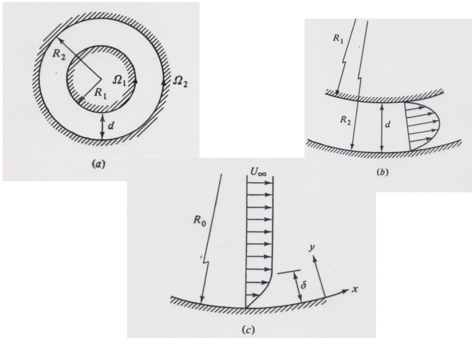
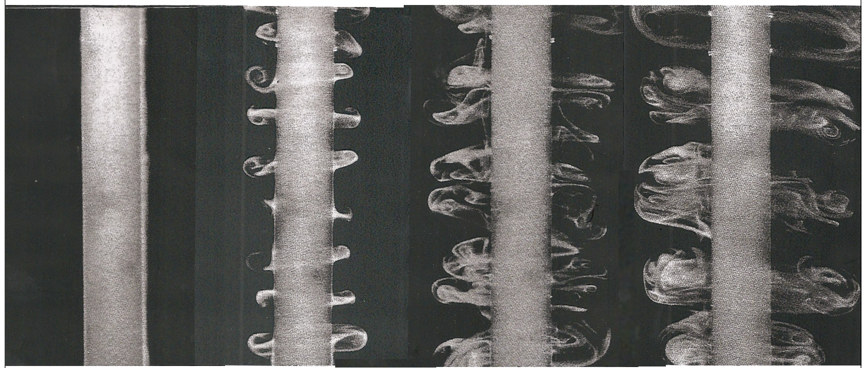
Centrifugal instability



Drazin & Reid, 1981



Rotating cylinder (large gap)

Centrifugal Instability

Let us consider a non-viscous fluid (i.e. Euler equations) in cylindrical coordinates. Then the Euler equations are in cylindrical coordinates

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}.$$

Mass conservation is given by

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \qquad \textit{(continuity)}$$

Consider axisymmetry so that we have $\frac{\partial}{\partial\theta} = 0$. The equations then reduce to

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = 0$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}.$$

Mass conservation is now given by

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

Rayleigh (1916) discovered that the θ-component of these equations can be written in the form

$$\frac{D(ru_\theta)}{Dt} = 0 \qquad \text{~Kelvin's circulation theorem}$$

.

This means that angular momentum is conserved, i.e.

$$\frac{Dru_\theta}{Dt} = \frac{D\Omega r^2}{Dt} = 0 \qquad \text{where } u_\theta = \Omega r$$

Or for the energy we can write

$$\frac{D(ru_\theta)^2}{Dt} = \frac{DL^2}{Dt} \text{ with } L^2 = (\Omega r^2)^2$$

For stability we have, according to Rayleigh's energy (1916) argument :

$$\frac{DL^2}{Dt} > 0$$

Rayleigh's energy argument

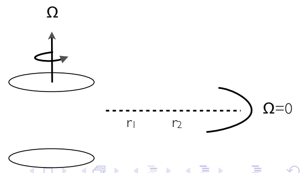
The potential energy is $\frac{\rho L^2}{2r^2} = \frac{1}{2}\frac{\Omega^2 r^4}{r^2} = \frac{\Omega^2 r^2}{2}$
the force F per unit mass in radial direction is :

$$F = \frac{u_\theta^2}{r} = \frac{L^2}{r^3}$$

in other words, there is a gradient force F with $\frac{dF}{dr} \neq 0$.
(Note, in a stratified fluid we have $F = mg$ and $\frac{dF}{dz} = \frac{dm}{dz}g \neq 0$)

Heuristic argument
Consider again the exchange of two rings of fluids (like before particles in a stratified fluid), supposing that their masses are equal :

$$2\pi r_1 dr_1 = 2\pi r_2 dr_2 = ds$$



when to expect instability ?

Consider the energy of two fluid rings r1 and r2,and there initial and final energy state ...
Initial state is :

$$E_{t_0} = \frac{L_1^2}{r_1^2} + \frac{L_2^2}{r_2^2} \text{ and } E_{t_1} = \frac{L_2^2}{r_1^2} + \frac{L_1^2}{r_2^2}$$

the difference in energy is then :

$$\Delta E = E_{t_1} - E_{t_0} = \left[(\frac{L_2^2}{r_1^2} + \frac{L_1^2}{r_2^2}) - (\frac{L_1^2}{r_1^2} + \frac{L_2^2}{r_2^2}) \right] ds$$

$$= (L_2^2 - L_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) ds$$

Instability when the kinetic energy increases ($\Delta E > 0$). thus stability for

$$\frac{DL^2}{Dt} > 0$$

this is known as Rayleigh's stability criterion for centrifugal instability.

For a displacement dr we have for stability when

$$\frac{\partial L^2/r^2}{\partial r} = \frac{\partial (r^2\Omega)^2}{\partial r} > 0$$

This criterion is often written in the form :

$$\phi(r) = \frac{1}{r^3}\frac{\partial (r^2\Omega)^2}{\partial r} > 0$$

this is a necessarry and sufficient condition !
In fact we have an angular 'stratification' of momentum, which is stable when constant or increasing monotonically with r.

→ NOTE

As for convection we can again estimate a characteristic number for centrifugal instability opposed by viscous effects. The Taylor number →

necessary and sufficient condition for *stability* ....

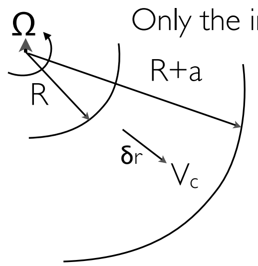
NOTE on Ri number:

The Richardson number Ri>0.25 is a necessary and sufficient condition for stability with respect to KH instability. (Abarbanel et al 1984, Ri>1 formally stable)

This means that an unstable shear flow is suppressed by buoyancy in strongly stratified flows, i.e. when the Richardson is high.

The condition for instability Ri<0.25 is necessary but not sufficient. For weak stratification, instability is set by the shear instability determined by the Rayleigh and Fjörtöft criteria; ——>.
These are only necessary conditions.

Taylor number



Only the inner cylinder rotates

Consider the Stokes effect on a little sphere of radius r_0, speed V_c and density ρ. the sphere looses momentum $p = mv_c$ in a time τ_c due to viscous drag, where mass $m = 4/3\pi r_0^3\rho$. Since the Stokes force due to friction on a particle is equal to $6\pi\mu r_0 v_c$ we obtain

$$m\frac{\partial v_c}{\partial t} = 4/3\pi r_0^3\rho\frac{\partial v_c}{\partial t} = \text{ Stokes force on the particle } = -6\pi\mu r_0 v_c$$

so that $\frac{1}{\tau_\nu} = \frac{1}{v_c}\frac{\partial v_c}{\partial t} = A\frac{\nu}{r_0^2}$, and A is a geometric constant.
After a time τ_ν the particle has travelled a distance $\delta_r = v_c\tau_\nu$.

Taylor number

The centrifugal force on the particle is $F_m = m\Omega^2(r)r$ and changes radially. Thus over the distance δ_r its variation is

$$F_m = m\frac{\partial\Omega^2(r)r}{\partial r}\delta r \approx Cm\frac{\Omega^2 R}{a}v_c\tau_\nu = C' r_0^3\rho\frac{\Omega^2 R}{a}v_c\tau_\nu = \frac{C'}{A}\frac{\rho}{\nu}\frac{v_c\Omega^2 r_0^5 R}{a}$$

where C' and A are geometric constants.
Instability for $F_m > F_{viscous}$, i.e. $\frac{C'}{A}\frac{\rho}{\nu}v_c\Omega^2\frac{r_0^5 R}{a} > 6\pi\rho\nu r_0 v_c$
so that

$$\frac{F_M}{F_{visc}} = \frac{\Omega^2 R r_0^4}{\nu^2 a} > \frac{A}{C'}6\pi$$

Neglecting geometric coefficients and with in the limit of the largest perturbation $r_0 \to a$, the Taylor number is

$$T = \frac{F_M}{F_{visc}} = \frac{\Omega^2 R\, a^3}{\nu^2}$$

(see Guyon, Hulin & Petit p 582)

Linear stability analyses of the inviscid case

Euler equations in cylindrical coord's

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}$$
$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta u_z$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}$$

Mass conservation is again given by

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0$$

The basic flow is only in azimuthal direction and given by $\vec{u} = (0, V, 0)$. (ρ=constant). Perturbations are given by

$$\begin{aligned} \vec{u} &= (u_r', V + u_\theta', u_z') \text{ and } V = \Omega r \\ p &= P + p', \end{aligned}$$

Substitution yields, after selection of the leading order terms, the perturbations equations (primes omitted)

$$\begin{aligned} \frac{\partial u_r}{\partial t} + \Omega \frac{\partial u_r}{\partial \theta} - \frac{2V}{r} u_\theta &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{\partial u_\theta}{\partial t} + \Omega \frac{\partial u_\theta}{\partial \theta} + \left(\frac{dV}{dr} + \frac{V}{r}\right) u_r &= -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \\ \frac{\partial u_z}{\partial t} + \Omega \frac{\partial u_z}{\partial \theta} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}$$

and continuity

$$\nabla . \vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

and $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

Consider perturbations of the form (see Drazin & Reid chapter 3, ; Chandrasekhar p 303) with $\frac{p}{\rho} = \varpi$, and ρ=constant.

$$(\vec{u}, \varpi) = (\hat{u}(r), \hat{v}(r), \hat{w}(r), \hat{\varpi}(r)) e^{(i(n\theta + kz) + \sigma t)}$$

$$\begin{aligned} (\sigma + in\Omega)u - \frac{2V}{r} v &= -D\varpi \\ (\sigma + in\Omega)v + (D_* V)u &= -i\frac{n}{r}\varpi \\ (\sigma + in\Omega)w &= -ik\varpi \\ D_* u + in\frac{v}{r} + ikw &= 0 \end{aligned}$$

with $D = \frac{d}{dr}$ and $D_* = \frac{d}{dr} + \frac{1}{r}$. Eliminating $v, w,$ and ϖ yields a single equation with $\gamma = \sigma + in\Omega$, $\phi(r)$ the Rayleigh determinant.

$$\gamma D \left(\frac{r^2 D_* u}{n^2 + k^2 r^2}\right) - \left\{\gamma^2 + \frac{k^2 r^2 \phi}{n^2 + k^2 r^2} + in\gamma r D \left(\frac{D_* V}{n^2 + k^2 r^2}\right)\right\} u = 0$$

Solve with $u = 0$ at $r = R_1$ and R_2.

In the *axisymmetric case*, the perturbations are the same for every value of θ since for all variables $\frac{\partial}{\partial \theta} = 0$.
But in the *non axisymmetric case*, we have to follow the perturbations on a particle motion along its azimuthal trajectory : we have to consider its **Lagrangian displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$** .
For a perturbation Δq on a basic flow Q (q=whatever) we can write

$$\Delta q = q(\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t), t) - Q(\boldsymbol{x}, t)$$

The corresponding Eulerian change is $q(\boldsymbol{x}, t) = Q(\boldsymbol{x}, t) + q'$.
To leading order in $\boldsymbol{\xi}$ this is

$$\Delta q = q' + \boldsymbol{\xi} . \boldsymbol{\nabla} q.$$

For an element at $\boldsymbol{x}$ the change in *Lagrangian velocity* at $\boldsymbol{x} + \boldsymbol{\xi}$ is

$$\Delta u = \frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{U} . \boldsymbol{\nabla} \boldsymbol{\xi}.$$

For the Eulerian change in velocity this implies

$$\boldsymbol{u}' = \frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{U} . \boldsymbol{\nabla} \boldsymbol{\xi} - \boldsymbol{\xi} . \boldsymbol{\nabla} \boldsymbol{U} \qquad (1)$$

while $\boldsymbol{\nabla} . \boldsymbol{u} = 0$ and $\boldsymbol{\nabla} . \boldsymbol{\xi} = 0$ by continuity.

Using the normal mode analyses we can set

$$\boldsymbol{\xi} = (\xi, \eta, \zeta) e^{(i(n\theta + kz) + \sigma t)}$$

so that with (1) follows $u = \gamma \xi, \quad v = \gamma \eta - r(D\Omega)\xi, \quad w = \gamma \zeta$

Substitution yields :

$$\begin{aligned} (\gamma^2 + 2r\Omega D\Omega)\xi - 2\Omega\gamma\eta &= -D\varpi \\ \gamma^2 \eta + 2\Omega\gamma\xi &= -i\frac{n}{r}\varpi \\ \gamma^2 \zeta &= -ik\varpi \end{aligned}$$

with continuity :

$$D_* \xi + in\frac{\eta}{r} + ik\zeta = 0.$$

With $\xi = 0$ at $r = R_1$ and R_2, and Ω=constant

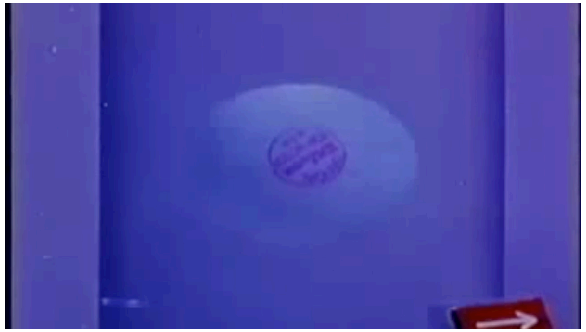
$$\left(D_* D - \frac{n^2}{r^2}\right)\varpi = k^2 \left(1 + \frac{4\Omega^2}{\gamma^2}\right)\varpi$$

and BC's :$D\varpi + \frac{2in\Omega}{\gamma r}\varpi = 0$ and $r = R_1$ and $r = R_2$.

$$\left(D_* D - \frac{n^2}{r^2}\right)\varpi = k^2\left(1 + \frac{4\Omega^2}{\gamma^2}\right)\varpi$$

and BC's : $D\varpi + \frac{2in\Omega}{\gamma r}\varpi = 0$ and $r = R_1$ and $r = R_2$.

This is the eigenvalue problem for the frequencies of a rotating column of fluid ! (Note that Ω= constant) (Kelvin 1880a, Chandrasekhar, Fultz 1964).



Related work is on the instability of vortices ($\Omega\neq$ constant) with a different eigenvalue problem and boundary conditions.

According to Drazin and Reid (refs to Howard 1962, and Howard and Gupta 1962) the Rayleigh criterion is invalid for azimuthal perturbations.

Later work shows that there are cases of validity of an adapted Rayleigh criterion.

Intermezzo vortices

Rayleigh's inflection point instability for circular flows

Consider two-dimensional perturbations $(u'_r, u'_\theta) \sim \phi(r)e^{(\sigma t + in\theta)}$ and no z' dependence (also no axi-symmetry) then we obtain for the dispersion relation

$$(\sigma + in\Omega)\left(D_* D - \frac{n^2}{r^2}\right)\phi - \frac{in}{r}\frac{dZ}{dr}\phi = 0 \qquad (3)$$

and $\frac{dZ}{dr} = r\frac{d^2\Omega}{dr^2} + 3\frac{d\Omega}{dr}$. From eq (3) we can derive Rayleigh's inflection point theorem as for a bounded shear flow. The flow is unstable when there is a change in sign of vorticity, i.e. there is an inflection point in the azimuthal velocity u'_θ.

⟶ Example of a vortex in a fluid in rigid body rotation

Exercise:

In a rotating fluid the Euler equations of motions ($\underline{u}$=(u,v,w)=(u_r,v_θ,w_z)) are given by

$$\frac{Du}{Dt} + fv - \frac{u^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$\frac{Dv}{Dt} + fu + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

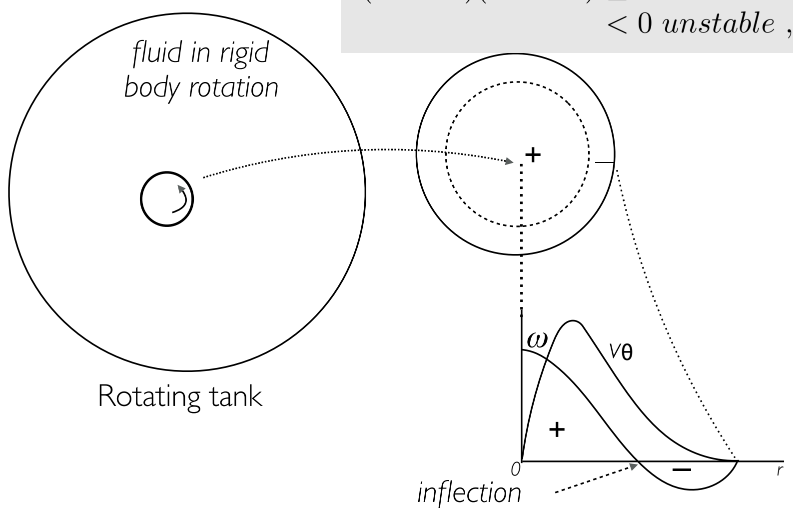
with D/Dt and continuity as above

Using the argument changing fluid rings, and *supposing axi-symmetry*, show that Rayleigh's criterion for instability reads:

$$\frac{d}{dr}(v_0 r + \frac{1}{2}fr^2)^2 < 0$$

Intermezzo vortices

$$(v + \Omega r)(\omega + 2\Omega) \geq 0 \ stable$$
$$< 0 \ unstable \ ,$$



stability if $v_{abs}\omega_{abs} > 0$ at all positions r in the vortex flow

Intermezzo vortices

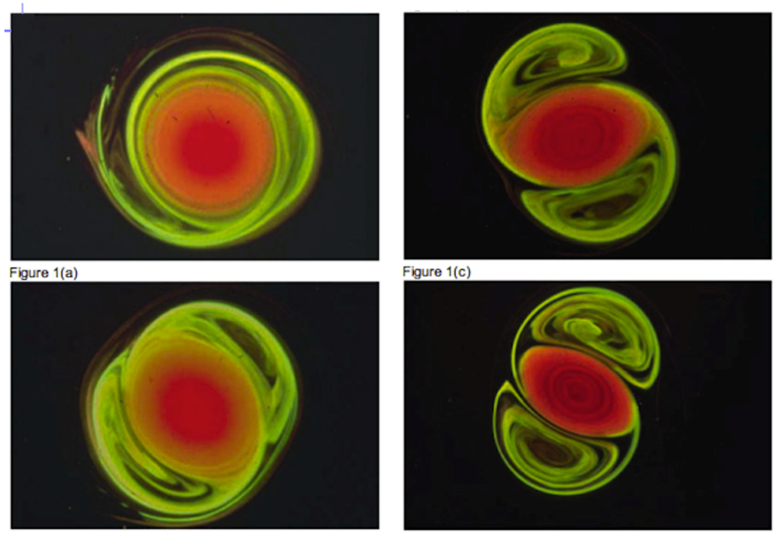


Figure 1(a) Figure 1(c) Figure 1(b) Figure 1(d)

van Heijst & Kloosterziel Nature 1991, Stability: Kloosterziel van Heijst JFM 1991

In a rotating system that rotates with angular velocity Ω – or on an f-plane – the equation for the azimuthal velocity of *circularly symmetric* flows reads

$$\frac{Dv}{Dt} + fu + \frac{uv}{r} = 0, \tag{4}$$

where u is, as usual, the radial velocity component, v the azimuthal component and f the Coriolis parameter. For a rotating tank $f = 2\Omega$. The material derivative is defined here as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z},$$

where w is the vertical v… (4) implies the following conservation law:

$$\frac{D}{Dt}(vr + \tfrac{1}{2}fr^2) = 0. \tag{5}$$

For a vortex located at the centre of the rotating tank the term within the brackets is the absolute angular momentum, or circulation, of a revolving fluid element at radius r. The equation for the radial velocity component is

$$\frac{Du}{Dt} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{6}$$

If the stationary basic vortex whose stability is under study has an azimuthal velocity distribution $v_0(r)$, the pressure-gradient force is necessarily

$$\frac{1}{\rho}\frac{dp_0}{dr} = \frac{v_0^2}{r} + fv_0. \tag{7}$$

If a fluid element is imagined to change its position slightly, from, say, r_0 to $r' = r_0 + \delta r$, it will acquire an azimuthal velocity $v'(r')$ that is determined by the conservation law expressed by (5):

$$v'(r')\,r' + \tfrac{1}{2}fr'^2 = v_0(r_0)\,r_0 + \tfrac{1}{2}fr_0^2. \tag{8}$$

This holds only for flows that are axisymmetric, and axisymmetry can only hold when all motion takes place in the form of an exchange of rings; this necessarily involves three-dimensional *overturning* motions.

Assuming that the prevailing pressure field is not changed by the motion, the element experiences an acceleration

$$\frac{D^2\delta r}{Dt^2} = \left\{\frac{v'^2}{r'} + fv'\right\} - \left\{\frac{v_0^2}{r'} + fv_0\right\}, \tag{9}$$

where v_0 is understood to be evaluated at $r = r'$. Taking (8) into account, the right-hand side is found to be equal to

$$\frac{1}{r'^3}\{(v_0(r_0)\,r_0 + \tfrac{1}{2}fr_0^2)^2 - (v_0(r')\,r' + \tfrac{1}{2}fr'^2)^2\}.$$

If this is developed in a Taylor series around $r = r_0$, one obtains

$$\frac{D^2\delta r}{Dt^2} = -\frac{\delta r}{r_0^3}\frac{d}{dr}(v_0\,r + \tfrac{1}{2}fr^2)^2|_{r_0} + O(\delta r^2). \tag{10}$$

Assume, for example, that δr is positive ($\delta r = u\delta t; u > 0$), then this equation tells us that there is a tendency to accelerate it even farther away from its original position if, for some r_0,

$$\frac{d}{dr}(v_0\,r + \tfrac{1}{2}fr^2)^2 < 0. \tag{11}$$

Taylor Couette: the basic state **viscous flow**

Assume an axisymmetric basic flow, i.e. :

$$u_\theta = V(r) = r\Omega(r) \text{ and } u_r = u_z = 0$$
$$p = P(r) \text{ and further } \frac{\partial}{\partial\theta} = 0$$

with the Navier Stokes equations we can solve the flow in the basic state

$$\frac{V^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r} \qquad (2)$$
$$0 = \nu(\Delta V - \frac{V}{r^2}) \qquad (3)$$

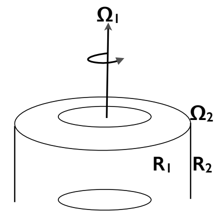
with the latter equation we obtain :

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} = 0$$

and thus $V = Ar + \frac{B}{r}$ or $\Omega = A + \frac{B}{r^2}$

Now solve for A and B.

Inviscid case:



Use boundary conditions : No slip conditions at R_1 and R_2

$$\Omega_1 = A + B/R_1^2$$
$$\Omega_2 = A + B/R_2^2$$

with $\mu = \frac{\Omega_2}{\Omega_1}$ and $\eta = \frac{R_1}{R_2}$ we obtain expressions for A and B

$$A = \Omega_1\eta^2\frac{1-\mu/\eta^2}{1-\eta^2} \quad \text{and} \quad B = \frac{\Omega_1 R_1^2(1-\mu)}{1-\eta^2}$$

Inviscid case:

The Rayleigh discriminant is defined as :

$$\Phi(r) \equiv \frac{1}{r^3}\frac{\partial(r^2\Omega)^2}{\partial r} > 0$$

(remember, this is the Rayleigh criterion for centrifugal stability)

so that with $\Omega = A + \frac{B}{r^2}$ we obtain $\Phi = 4A(A + \frac{B}{r^2})$, and the definitions of A and B we get :

$$\phi = -4\Omega^2\eta^4\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)}^2\left(\frac{1}{r^2}-\kappa\right)$$

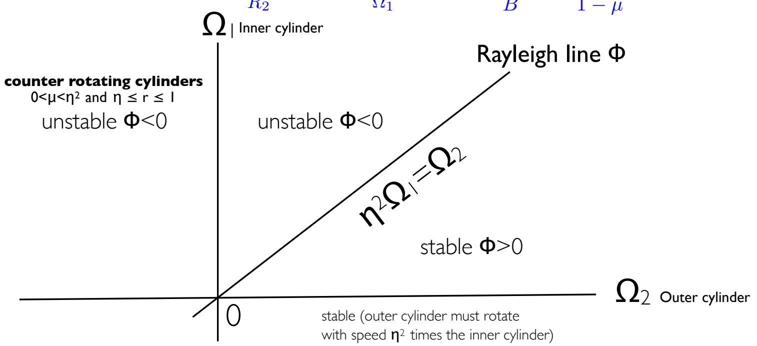
where $\kappa = -\frac{AR_1^2}{B} = \frac{1-\mu/\eta^2}{1-\mu}$.

With ϕ we can determine stability of the Taylor Couette flow.

Inviscid case:

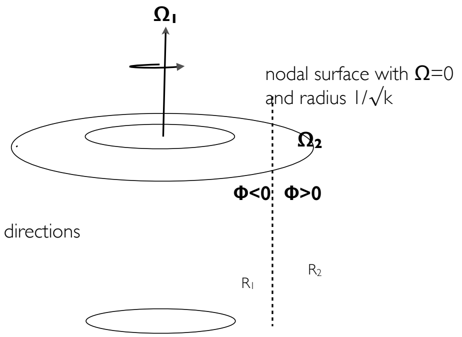
$$\phi = -4\Omega_1^2\eta^4\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)}^2\left(\frac{1}{r^2}-\kappa\right)$$

$$\eta = \frac{R_1}{R_2} \qquad \mu = \frac{\Omega_2}{\Omega_1} \qquad \kappa = -\frac{AR_1^2}{B} = \frac{1-\mu/\eta^2}{1-\mu}$$



Viscous effects will postpone the instability, i.e. in the viscous case $\phi_c > \phi_{c\ inviscid}$

Inviscid case:



$\Phi<0$
cylinders rotate in opposite directions
(μ<0)

Taylor number

$$Ta = \frac{4\Omega_1^2}{\nu^2} R_1^4 \frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)^2}$$

$$\eta = \frac{R_1}{R_2} \qquad \mu = \frac{\Omega_2}{\Omega_1}$$

Taylor number for Taylor-Couette flow
viscous effects postpone instability
Ta~ centrifugal/viscous

Remember Re ~ inertia/viscous effects

Only the inner cylinder rotates : $\mu=0$, $k=1$ and $Ta=...R_1^4/(1-\eta^2)^2 \approx 4\Omega_1^2 R_1^4/\nu^2$

Exercise : The Rayleigh discriminant is defined as $\Phi(r) = \frac{1}{r^3}\frac{d(r^2\Omega)^2}{dr}$.
Show that for $\Phi > 0 \Rightarrow k^2/\sigma^2 < 0$ and $\Phi < 0 \Rightarrow k^2/\sigma^2 > 0$.

Linear stability analyses of the **viscous** Taylor problem

The basic flow is only in azimuthal direction and given by $\vec{u} = (0, V, 0)$. (ρ=constant). Perturbations are given by

$$\vec{u} = (u_r, V + u_\theta, u_z)$$
$$p = P + \delta p, \quad \text{and} \quad \varpi = \frac{\delta p}{\rho}$$

Substitution yields, after selection of the leading order terms, the perturbations equations

$$\frac{\partial u_r}{\partial t} - \frac{2V}{r} u_\theta = -\frac{\partial \varpi}{\partial r} + \nu(\nabla^2 u_r - \frac{u_r}{r^2})$$
$$\frac{\partial u_\theta}{\partial t} + \left(\frac{dV}{dr} + \frac{V}{r}\right) u_r = \nu(\nabla^2 u_\theta - \frac{u_\theta}{r^2})$$
$$\frac{\partial u_z}{\partial t} = -\frac{\partial \varpi}{\partial z} + \nu \nabla^2 u_z$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

and continuity gives $\nabla.\vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$

Linear stability analyses of the viscous Taylor problem

Consider perturbations of the form (see Drazin & Reid p.91-93, ; Chandrasekhar p 303)

$$(\vec{u}, \varpi) = (\hat{u}(r), \hat{v}(r), \hat{w}(r), \hat{\varpi}(r) e^{(ikz+\sigma t)}$$

$$\sigma u - \frac{2V}{r} v = -D\varpi + \nu(DD^* v - k^2)$$
$$\sigma v + (D_* V) u = \nu(DD^* - k^2) u$$
$$\sigma w = -ik\varpi +$$
$$D_* u + ikw = 0$$

With $D = \frac{d}{dr}$ and $D_* = \frac{d}{dr} + \frac{1}{r}$ we obtain :

$$[\nu(DD_* - k^2) - \sigma](DD_* - k^2)\, u = 2k^2\Omega\, v$$
$$[\nu(DD_* - k^2) - \sigma]\, v = (D_* V)\, u$$

With

$$x = (r - R_0)/d \text{ where } R_0 = \frac{1}{2}(R_1 + R_2), \quad \Omega(r) = \Omega_1 g(x)$$

$$a = kd \text{ and } \omega = \sigma d^2/\nu, \text{ and replace } \quad u\, 2Ad^2/\nu \to u$$

we obtain in dimensionless units after substitution ...

$$(DD_* - a^2 - \omega)(DD_* - a^2)\, u = -a^2 T\, g(x)\, v$$

$$(DD_* - a^2 - \omega)\, v = u$$

$$\text{and } u = Du = v = 0 \text{ at } x = \pm\frac{1}{2}$$

$$T = \frac{-4A\Omega_1 d^4}{\nu^2} = \frac{-4A\Omega_1 R_1^4}{\nu^2}\, f(\eta, \mu)$$

is the Taylor number, f a function of $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2$.

Solutions have been found for the narrow gap approximation :

$$d = R_2 - R_1 \ll \frac{1}{2}(R_1 + R_2)$$

The approximate Taylor number is then $T = \frac{4A\Omega_1 d^4}{\nu}$

Consequences :

1) $D_* = D$
2) $\Omega(r) \sim \Omega_1[1 - (1-\mu)\frac{r - R_1}{d}]$, i.e. a linear variation with r
3) Cross-term perturbations in the centrifugal term ($\sim u_r u_\theta / r$) neglected.

In the marginal state, $\omega = 0$ (i.e. $\sigma = 0$), the general equations reduce to

$$(D^2 - a^2)^3 u = -a^2 T \left(1 + \alpha(x + \frac{1}{2})\right) v \quad \text{and } \alpha = \mu - 1$$

$$(D^2 - a^2)\, v = u$$

$$u = Du = v = 0 \text{ at } x = \pm\frac{1}{2}$$

Solutions are of the form $\quad v = \Sigma_{m=1}^{\infty} C_m \sin m\pi x$

Substitution of this solution in the equations allows us to find the general solution

$$u = \Sigma_m^{\infty} \frac{C_m}{(m^2\pi^2 + a^2)^2} \{A_1^m \cosh ax + B_1^m \sinh ax + ...\}$$

The coefficients A_1^m, A_2^m and B_1^m, B_2^m can be determined with the boundary conditions $u = Du = v = 0$ at $x = \pm\frac{1}{2}$.

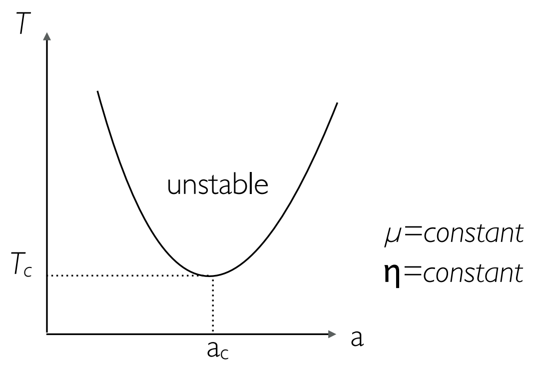
Manipulation of the thus obtained equations provides to leading order the solution

$$T = \frac{2}{2+\alpha} \frac{(\pi^2 + a^2)^3}{a^2\{1 - 16a\pi^2 \cosh^2 \frac{1}{2}a/[(\pi^2 + a^2)^2(sinha + a)]\}}$$

In approximation ($\alpha = -(1-\mu)$) we obtain then for $a_{min} = 3.12$ the critical Taylor number equal to

$$T_c = \frac{2}{2+\alpha} 1715 = \frac{1715}{\frac{1}{2}(1+\mu)} \text{ with } 0 < \mu < 1$$

(1^{rst} approx. Drazin & Reid), More precise approximations (Chandrasekhar) obtain $T_c = 1708$.



Critical Taylor numbe T_c = 1708 and minimum wave number a=3.12.

Physics are analogue to convection and critical numbers are very close.

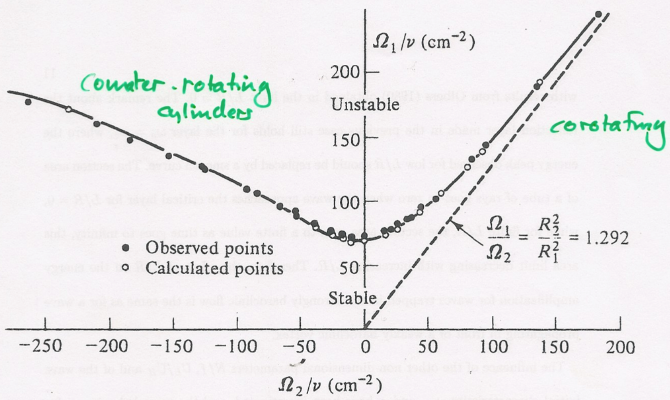
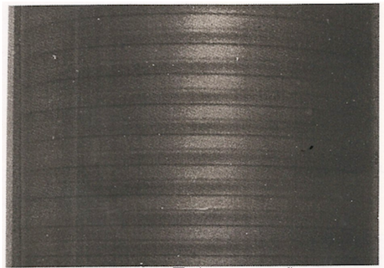
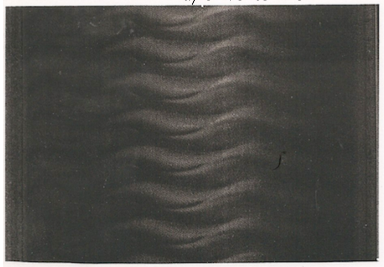


Fig. 3.13. Observations and narrow-gap calculations of the curve of marginal stability for water when $R_1 = 3.55$ cm and $R_2 = 4.035$ cm. (From Taylor 1923.)



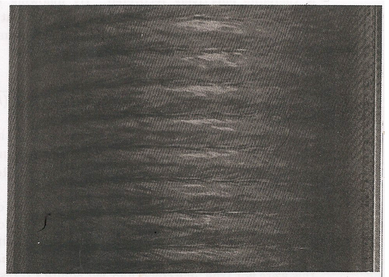
Taylor vortex flow



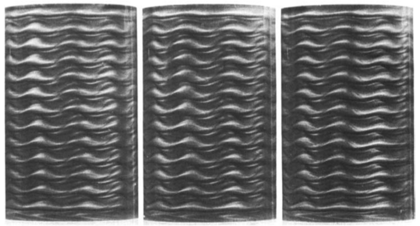
128. **Laminar Taylor vortices in a narrow gap.** A larger inner cylinder in the

wavy vortex flow

Turbulent vortex flow



131. **Axisymmetric turbulent Taylor vortices.** The conditions are as in the pair of photographs on the opposite page, but at 1625 times the critical speed. A sudden start produces chaotic motion at first, but this regular permanent turbulent pattern emerges within a minute. *Koschmieder 1979*



Modulated wavy flow

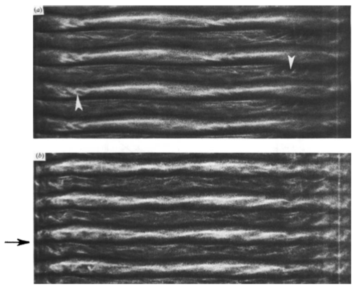


FIGURE 22. (*a*) The arrows point to the 'ripple' (RIP) observed in wavy-Taylor-vortex flow; $R_i = 1750$, $R_o = 250$. (*b*) The arrow points to the 'corkscrew' (CKS) pattern observed on wavy-Taylor-vortex-flow boundaries; $R_i = 2030$, $R_o = 300$. $\Gamma = 30$ in both cases.

Corkscrew patterns

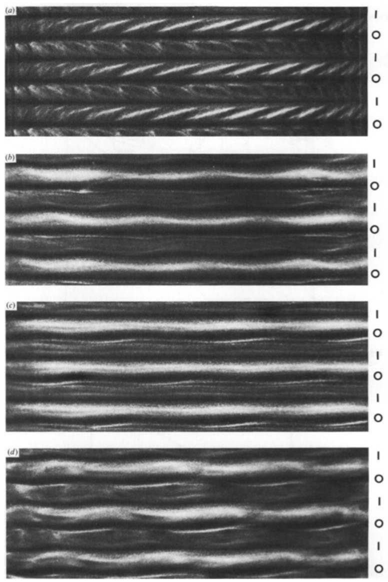
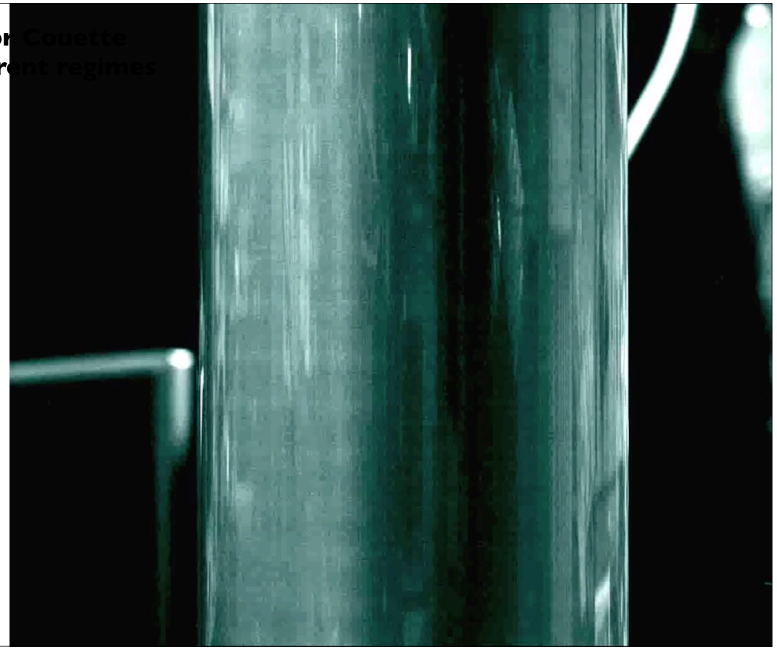


FIGURE 20. (*a*) Twisted Taylor vortices (TWI); $R_i = 1040$, $R_o = 720$. (*b*) Wavy inflow boundaries (WIB); $R_i = 1310$, $R_o = 700$. (*c*) Wavy outflow boundaries (WOB); $R_i = 1170$, $R_o = 700$. (*d*) Wavelets (WVL); $R_i = 1250$, $R_o = 730$. $\Gamma = 30$ for all four cases. The letters I and O indicate the inflow and outflow boundaries, respectively.

Taylor Couette
Different regimes



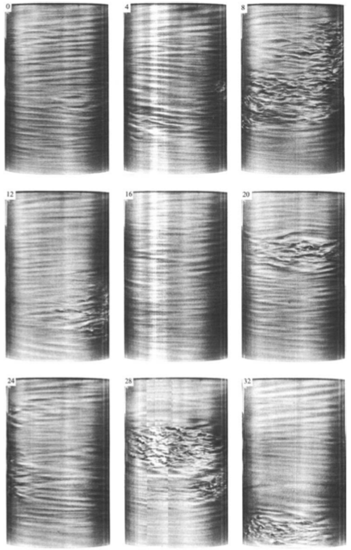


FIGURE 9. Intermittent turbulent spots (INT) in a flow with interpenetrating laminar spirals. The numbers labelling the photographs give the time in seconds; the period of the inner cylinder was 4.25 s, the outer cylinder, 1.87 s ($R_o = -1500$, $R_i = 590$).

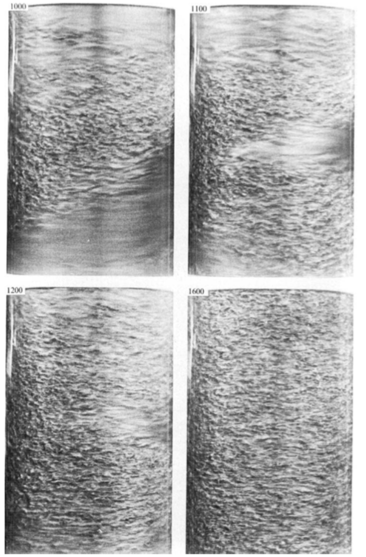


FIGURE 14. Photographic sequence of spiral turbulence (SPT) as R_i was increased for fixed $R_o = -3000$. The numbers on the photographs indicate R_i.

Intermittent turbulence

Spiral turbulence

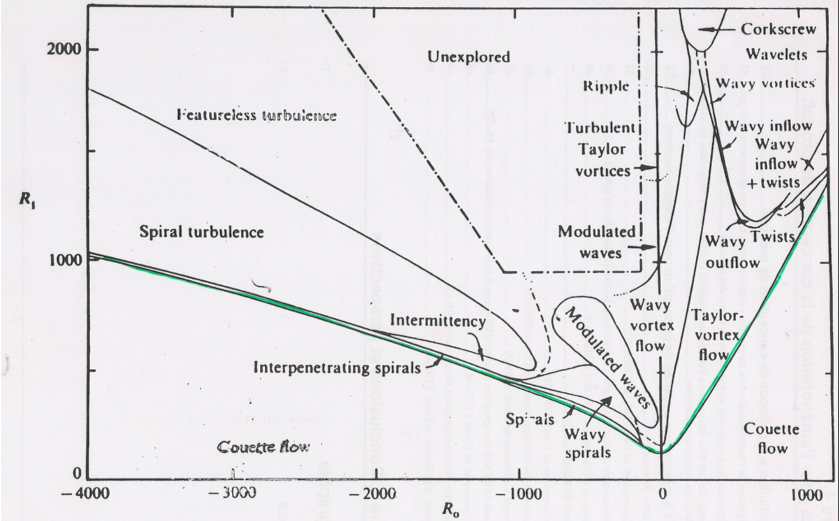
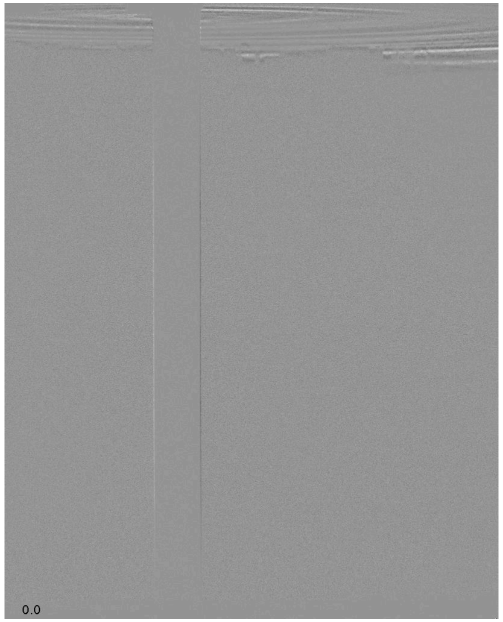


FIG. 17.12 Regime diagram for rotating Couette flow. (Note that the name 'Couette flow' on the diagram denotes the purely azimuthal motion, Section 9.3). From Ref. [61].

Andereck, Liu & Swinney JFM 164 1986

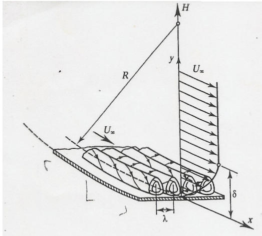
IN stratified fluids

Observations for Ω/N>1



Görtler vortices

Gortler vortices appear in curved boundaries due to the centrifugal force. The 'stratification' of the centrifugal force in the boundary is unstable.



$\delta \sim \sqrt{\nu L/U_\infty}$ and $\delta/L \sim Re^{-\frac{1}{2}}$. The approximation is close to that of the Taylor flow (therefore often called Taylor-Gortler vortices). For the stability analyses three approximations are made

1) The boundary layer is much thinner than the radius $\delta \ll R$. This corresponds to the thin gap approximation.
2) The basic flow is parallel to the boundary. The centrifugal effects only appear in the perturbations.
3) The stability analyses is local, and independent on x, whereas the y component of the basic flow is neglected

to leading order in δ/R we obtain

$$\begin{aligned}(D^2 - a^2 - \omega)(D^2 * - a^2)\, v &= -a^2 \mu U u \\ (D^2 - a^2 - \omega)\, u &= \frac{dU}{d\eta} u \\ \text{boundary conditions} : u = v = \frac{dv}{d\eta} &= 0 \text{ at } \eta = 0 \text{ and } \eta \to \infty\end{aligned}$$

$\eta = y/\delta$, $a = k\delta$ and $\omega = \sigma\delta^2/\nu$. $\mu = (2U_\infty\delta/\nu)^2(\delta/R)$.
The Görtler number is

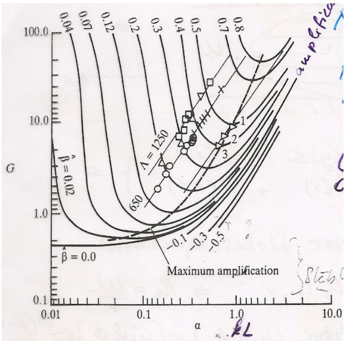
$$G = (U_\infty\theta/\nu)\,(\theta/R)^{\frac{1}{2}}$$

The Görtler number is based on the momentum thickness θ of the boundary layer, with

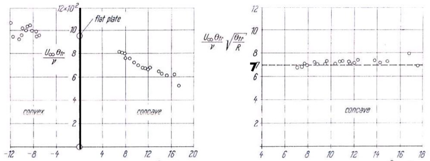
$$\theta = Constant\sqrt{\nu x/U_\infty} \sim \delta(x)$$

Constant ≈ 0.664 Schlichting 1960

Görtler vortices



$Re_c = (U_\infty \theta_{tr})/\nu$

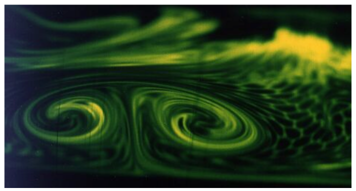


θ/R θ/R .10⁵

Fig. 17.33. Measurements on the point of transition of slightly concave walls, after H. W. Liepmann [67, 68]. (a) critical Reynolds number $\frac{U_\infty \theta_{tr}}{\nu}$ versus $\frac{\theta}{R}$; (b) the characteristic quantity $\frac{U_\infty \theta_{tr}}{\nu}\sqrt{\frac{\theta_{tr}}{R}}$ versus $\frac{\theta}{R}$
θ = momentum thickness; R = radius of curvature of wall

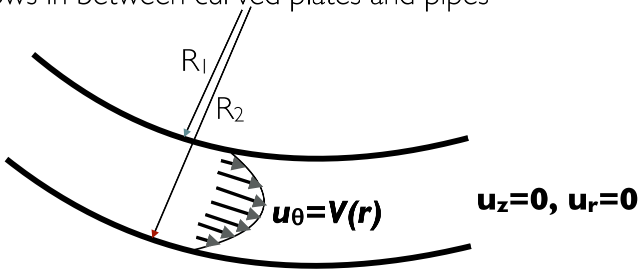
Schlichting 1960

θ_{tr} = momentum thickness at transition.



cross view of Görtler votices
PetitJean JFM 1994

Dean problem (1928)
for flows in between curved plates and pipes



N.S equations in cylindrical coord's *with Ur =Uz=0*

$$\begin{aligned}\frac{Du_r}{Dt} - \boxed{\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \\ \frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = \boxed{-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta u_\theta - \frac{u_\theta}{r^2}\right.} + \left.\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) \\ \frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta u_z\end{aligned}$$

two balances

DEAN problem

N.S equations in cylindrical coord's

$U_r = U_z = 0$

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta u_z$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

Mass conservation is again given by

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Thus directly from the Navier Stokes equation we have the two balances

$$\frac{V^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{and} \quad \nu DD_* V = \frac{1}{\rho r}\frac{\partial p}{\partial \theta}\Big|_0$$

The general solution of this flow is given by

$$V(r) = \frac{1}{2\rho\nu}\frac{\partial p}{\partial \theta}\Big|_0 \left(r\ln(r) + Cr + \frac{E}{r}\right)$$

and boundary conditions

$$V(R_1) = V(R_2) = 0$$

providing expressions for $E = f((R_1, R_2)$ and $C = g(R_1, R_2)$.

Dean instability

- Small gap approximation :

d<<R₀ where $d = R_2 - R_1$ and $R_0 = (R_1 + R_1)/2$

$x = \overline{r - R_0}/d$ and $-1/2 \leq x \leq 1/2$

$\Longrightarrow V(r) \approx 3/2\, V_m\,(1 - 4x^2)$ $\quad V_m = \frac{-d^2}{12\rho\nu^2 R_1}\frac{\partial P}{\partial \theta}\Big|_0$

Note: Rayleigh centrifugal criterion (instability) $\frac{d}{dr}(r^2\Omega)^2 < 0$

Flow is unstable for $0 < x < 1/2$

Flow is table for $-1/2 < x < 0$

Dean instability

- As for Taylor-Couette flow axisymmetric perturbations give

$$(D^2 - a^2 - \sigma)(D - a^2)u = (1 - 4x^2)v \qquad a = kd$$

$$(D^2 - a^2 - \sigma)v = -a^2\lambda x u \qquad \sigma = \frac{sd^2}{\nu}$$

With boundary conditions $Du = u = v = 0$ at $x = \pm\frac{1}{2}$

$$\Lambda = \frac{36Re^2 d}{R_1} \quad \text{and} \quad Re = \frac{V_m d}{\nu}$$

Λ is equivalent to the Taylor number **Ta**

In the literature, the Dean number is used $De = Re\left(\frac{d}{R_1}\right)^{\frac{1}{2}}$

Critical values for onset of instability are

$\Lambda_c = 46458$ and $De = 35.92$ for $a_c = 3.95$.

(Gibson & Cook 1974, numerical solutions)

Dean instability

Exercise:

Compare this with a Poiseuille flow perturbed with a 2D disturbance which is unstable for

$$\frac{3}{2}V_m\left(\frac{d}{2}\right)\frac{1}{\nu} > 5772 \quad \text{i.e. when} \quad Re > Re_c = 7696$$

How straight should a canal be to see this instability and NOT the Dean instability ?